COARSE REDUCIBILITY AND ALGORITHMIC RANDOMNESS

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ABSTRACT. A *coarse description* of a set $A \subseteq \omega$ is a set $D \subseteq \omega$ such that the symmetric difference of A and D has asymptotic density 0. We study the extent to which noncomputable information can be effectively recovered from all coarse descriptions of a given set A, especially when A is effectively random in some sense. We show that if A is 1-random and B is computable from every coarse description D of A, then B is K-trivial, which implies that if A is in fact weakly 2-random then B is computable. Our main tool is a kind of compactness theorem for cone-avoiding descriptions, which also allows us to prove the same result for 1-genericity in place of weak 2-randomness. In the other direction, we show that if $A \leqslant_{\mathrm{T}} \emptyset'$ is a 1-random set, then there is a noncomputable c.e. set computable from every coarse description of A, but that not all K-trivial sets are computable from every coarse description of some 1-random set. We study both uniform and nonuniform notions of coarse reducibility. A set Y is *uniformly coarsely reducible* to X if there is a Turing functional Φ such that if D is a coarse description of X, then Φ^D is a coarse description of Y. A set B is *nonuniformly coarsely reducible* to A if every coarse description of A computes a coarse description of B. We show that a certain natural embedding of the Turing degrees into the coarse degrees (both uniform and nonuniform) is not surjective. We also show that if two sets are mutually weakly 3-random, then their coarse degrees form a minimal pair, in both the uniform and nonuniform cases, but that the same is not true of every pair of relatively 2-random sets, at least in the nonuniform coarse degrees.

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1. Introduction

There are many natural problems with high worst-case complexity that are nevertheless easy to solve in most instances. The notion of "generic-case complexity" was introduced by Kapovich, Myasnikov, Schupp, and Shpilrain [14] as a notion that is more tractable than average-case complexity but still allows a somewhat nuanced analysis of such problems. That paper also introduced the idea of generic computability, which captures the idea of having a partial algorithm that correctly computes $A(n)$ for "almost all" n, while never giving an incorrect answer. Jockusch and Schupp [13] began the general computability theoretic investigation of generic computability and also defined the idea of coarse computability, which captures the idea of having a total algorithm that always answers and may make mistakes, but correctly computes $A(n)$ for "almost all" n. We are here concerned with this latter concept. We first need a good notion of "almost all" natural numbers.

Definition 1.1. Let $A \subseteq \omega$. The *density of A below n*, denoted by $\rho_n(A)$, is $\frac{|A\upharpoonright n|}{n}$. The *upper density* $\overline{\rho}(A)$ of A is $\limsup_n \rho_n(A)$. The *lower density* $\underline{\rho}(A)$ of A is $\liminf_n \rho_n(A)$. If $\overline{\rho}(A) = \underline{\rho}(A)$ then we call this quantity the *density* of A, and denote it by $\rho(A)$.

We say that D is a *coarse description* of X if $\rho(D \triangle X) = 0$, where $\triangle$ denotes symmetric difference. A set X is *coarsely computable* if it has a computable coarse description.

This idea leads to natural notions of reducibility.

Definition 1.2. We say that Y is *uniformly coarsely reducible* to X, and write $Y \leqslant_{\mathrm{uc}} X$, if there is a Turing functional Φ such that if D is a coarse description of X, then Φ^D is a coarse description of Y. This reducibility induces an equivalence relation $\equiv_{\mathrm{uc}}$ on 2^ω. We call the equivalence class of X under this relation the *uniform coarse degree* of X.

Uniform coarse reducibility, generic reducibility (defined in [13]), and several related reducibilities have been termed *notions of robust information coding* by Dzhafarov and Igusa [7]. Work on such notions has mainly focused on their uniform versions. (One exception is a result on nonuniform ii-reducibility in Hirschfeldt and Jockusch [9].) However, their nonuniform versions also seem to be of interest. In particular, we will work with the following nonuniform version of coarse reducibility.

Definition 1.3. We say that Y is *nonuniformly coarsely reducible* to X, and write $Y \leqslant_{\mathrm{nc}} X$, if every coarse description of X computes a coarse description of Y. This reducibility induces an equivalence relation $\equiv_{\mathrm{nc}}$ on 2^ω. We call the equivalence class of X under this relation the *nonuniform coarse degree* of X.

Note that the coarsely computable sets form the least degree in both the uniform and nonuniform coarse degrees. Uniform coarse reducibility clearly implies nonuniform coarse reducibility. We will show in the next section that, as one might expect, the converse fails. The development of the theory of notions of robust information coding and related concepts have led to interactions with computability theory (as in Jockusch and Schupp [13]; Downey, Jockusch, and Schupp [4]; Downey, Jockusch, McNicholl, and Schupp [5]; and Hirschfeldt, Jockusch, McNicholl, and Schupp [10]), reverse mathematics (as in Dzhafarov and Igusa [7] and Hirschfeldt and Jockusch [9]), and algorithmic randomness (as in Astor [1]).

In this paper, we investigate connections between coarse reducibility and algorithmic randomness. In Section 2, we describe natural embeddings of the Turing degrees into the uniform and nonuniform coarse degrees, and discuss some of their basic properties. In Section 3, we show that no weakly 2-random set can be in the images of these embeddings by showing that if X is weakly 2-random and A is noncomputable, then there is some coarse description of X that does not compute A. More generally, we show that if X is 1-random and A is computable from every coarse description of X, then A is K-trivial. Our main tool is a kind of compactness theorem for cone-avoiding descriptions. We also show that there do exist noncomputable sets computable from every coarse description of some 1-random set, but that not all K-trivial sets have this property. In Section 4, we give further examples of classes of sets that cannot be in the images of our embeddings. In Section 5, we show that if two sets are relatively weakly 3-random then their coarse degrees form a minimal pair, in both the uniform and nonuniform cases, but that, at least for the nonuniform coarse degrees, the same is not true of every pair of relatively 2-random sets. These results are analogous to the fact that, for the Turing degrees, two relatively weakly 2-random sets always form a minimal pair, but two relatively 1-random sets may not. In Section 6, we conclude with a few open questions.

We assume familiarity with basic notions of computability theory (as in [22]) and algorithmic randomness (as in [3] or [19]). For $S \subseteq 2^{<\omega}$, we write $[\![S]\!]$ for the open subset of 2^ω generated by S; that is, $[\![S]\!] = \{X : \exists n\, (X \upharpoonright n \in S)\}$. We denote the uniform measure on 2^ω by μ.

2. Coarsenings and embeddings of the Turing degrees

We can embed the Turing degrees into both the uniform and nonuniform coarse degrees, and our first connection between coarse computability and algorithmic randomness comes from considering such embeddings. While there may be several ways to define such embeddings, a natural way to proceed is to define a map $\mathcal{C} : 2^\omega \to 2^\omega$ such that $\mathcal{C}(A)$ contains the same information as A, but coded in a "coarsely robust" way. That is, we would like $\mathcal{C}(A)$ to be computable from A, and A to be computable from any coarse description of $\mathcal{C}(A)$.

In the case of the uniform coarse degrees, one might think that the latter reduction should be uniform, but that condition would be too strong: If $\Gamma^D = A$ for every coarse description D of $\mathcal{C}(A)$ then $\Gamma^\sigma(n)\downarrow \Rightarrow \Gamma^\sigma(n) = A(n)$ (since every string can be extended to a coarse description of $\mathcal{C}(A)$), which, together with the fact that for each n there is a σ such that $\Gamma^\sigma(n)\downarrow$, implies that A is computable. Thus we relax the uniformity condition slightly in the following definition.

Definition 2.1. A map $\mathcal{C} : 2^\omega \to 2^\omega$ is a *coarsening* if for each A we have $\mathcal{C}(A) \leqslant_{\mathrm{T}} A$, and for each coarse description D of $\mathcal{C}(A)$, we have $A \leqslant_{\mathrm{T}} D$. A coarsening $\mathcal{C}$ is *uniform* if there is a binary Turing functional Γ with the following properties for every coarse description D of $\mathcal{C}(A)$:

1. Γ^D is total.
2. Let $A_s(n) = \Gamma^D(n, s)$. Then $A_s = A$ for cofinitely many s.

Proposition 2.2. *Let $\mathcal{C}$ and $\mathcal{F}$ be coarsenings and A and B be sets. Then*

1. *$B \leqslant_{\mathrm{T}} A$ if and only if $\mathcal{C}(B) \leqslant_{\mathrm{nc}} \mathcal{C}(A)$.*
2. *If $\mathcal{C}$ is uniform then $B \leqslant_{\mathrm{T}} A$ if and only if $\mathcal{C}(B) \leqslant_{\mathrm{uc}} \mathcal{C}(A)$.*
3. *$\mathcal{C}(A) \equiv_{\mathrm{nc}} \mathcal{F}(A)$, and*
4. *if $\mathcal{C}$ and $\mathcal{F}$ are both uniform then $\mathcal{C}(A) \equiv_{\mathrm{uc}} \mathcal{F}(A)$.*

Proof. 1. Suppose that $\mathcal{C}(B) \leqslant_{\mathrm{nc}} \mathcal{C}(A)$. Then $\mathcal{C}(A)$ computes a coarse description D_1 of $\mathcal{C}(B)$. Thus $B \leqslant_{\mathrm{T}} D_1 \leqslant_{\mathrm{T}} \mathcal{C}(A) \leqslant_{\mathrm{T}} A$.

Now suppose that $B \leqslant_{\mathrm{T}} A$ and let D_2 be a coarse description of $\mathcal{C}(A)$. Then $\mathcal{C}(B) \leqslant_{\mathrm{T}} B \leqslant_{\mathrm{T}} A \leqslant_{\mathrm{T}} D_2$. Thus $\mathcal{C}(B) \leqslant_{\mathrm{nc}} \mathcal{C}(A)$.

2. Suppose that $\mathcal{C}$ is uniform and that $B \leqslant_{\mathrm{T}} A$. Let D_2 be a coarse description of $\mathcal{C}(A)$. Let A_s be as in Definition 2.1, with $D = D_2$. Then $\mathcal{C}(B) \leqslant_{\mathrm{T}} B \leqslant_{\mathrm{T}} A$, so let Φ be such that $\Phi^A = \mathcal{C}(B)$. Let $X \leqslant_{\mathrm{T}} D_2$ be defined as follows. Given n, search for an $s > n$ such that $\Phi^{A_s}(n)\downarrow$ and let $X(n) = \Phi^{A_s}(n)$. (Note that such an s must exist.) Then $X(n) = \Phi^A(n) = \mathcal{C}(B)(n)$ for almost all n, so X is a coarse description of $\mathcal{C}(B)$. Since X is obtained uniformly from D_2, we have $\mathcal{C}(B) \leqslant_{\mathrm{uc}} \mathcal{C}(A)$. The converse follows immediately from 1.

3. Let D_3 be a coarse description of $\mathcal{F}(A)$. Then $\mathcal{C}(A) \leqslant_{\mathrm{T}} A \leqslant_{\mathrm{T}} D_3$. Thus $\mathcal{C}(A) \leqslant_{\mathrm{nc}} \mathcal{F}(A)$. By symmetry, $\mathcal{C}(A) \equiv_{\mathrm{nc}} \mathcal{F}(A)$.

4. If $\mathcal{F}$ is uniform then the same argument as in the proof of 2 shows that we can obtain a coarse description of $\mathcal{C}(A)$ uniformly from D_3, whence $\mathcal{C}(A) \leqslant_{\mathrm{uc}} \mathcal{F}(A)$. If $\mathcal{C}$ is also uniform then $\mathcal{C}(A) \equiv_{\mathrm{uc}} \mathcal{F}(A)$ by symmetry. □

Thus uniform coarsenings all induce the same natural embeddings. It remains to show that uniform coarsenings exist. One example is given by Dzhafarov and Igusa [7]. We give a similar example. Let $I_n = [n!, (n+1)!)$ and let $\mathcal{I}(A) = \bigcup_{n \in A} I_n$; this map first appeared in Jockusch and Schupp

[13]. Clearly $\mathcal{I}(A) \leqslant_{\mathrm{T}} A$, and it is easy to check that if D is a coarse description of $\mathcal{I}(A)$ then D computes A. Thus $\mathcal{I}$ is a coarsening.

To construct a uniform coarsening, let $\mathcal{H}(A) = \{\langle n, i\rangle : n \in A \ \wedge \ i \in \omega\}$ and define $\mathcal{E}(A) = \mathcal{I}(\mathcal{H}(A))$. The notation $\mathcal{E}$ denotes this particular coarsening throughout the paper.

Proposition 2.3. *The map $\mathcal{E}$ is a uniform coarsening.*

Proof. Clearly $\mathcal{E}(A) \leqslant_{\mathrm{T}} A$. Now let D be a coarse description of $\mathcal{E}(A)$. Let $G = \{m : |D \cap I_m| > \frac{|I_m|}{2}\}$ and let $A_s = \{n : \langle n, s\rangle \in G\}$. Then $G =^* \mathcal{H}(A)$, so $A_s = A$ for all but finitely many s, and the A_s are obtained uniformly from D. $\square$

A first natural question is whether uniform coarse reducibility and nonuniform coarse reducibility are indeed different. We give a positive answer by showing that, unlike in the nonuniform case, the mappings $\mathcal{E}$ and $\mathcal{I}$ are not equivalent up to uniform coarse reducibility. Recall that a set X is *autoreducible* if there exists a Turing functional Φ such that for every $n \in \omega$ we have $\Phi^{X \setminus \{n\}}(n) = X(n)$. Equivalently, we could require that Φ not ask whether its input belongs to its oracle. We now introduce a Δ^0_2-version of this notion.

Definition 2.4. A set X is *jump-autoreducible* if there exists a Turing functional Φ such that for every $n \in \omega$ we have $\Phi^{(X \setminus \{n\})'}(n) = X(n)$.

Proposition 2.5. *Let X be such that $\mathcal{E}(X) \leqslant_{\mathrm{uc}} \mathcal{I}(X)$. Then X is jump-autoreducible.*

Proof. We must give a procedure for computing $X(n)$ from $(X \setminus \{n\})'$ that is uniform in X. Given an oracle for $X \setminus \{n\}$, we can uniformly compute $\mathcal{I}(X \setminus \{n\})$. Now $\mathcal{I}(X \setminus \{n\}) =^* \mathcal{I}(X)$, so $\mathcal{I}(X \setminus \{n\})$ is a coarse description of $\mathcal{I}(X)$. Since $\mathcal{E}(X) \leqslant_{\mathrm{uc}} \mathcal{I}(X)$ by assumption, from $\mathcal{I}(X \setminus \{n\})$ we can uniformly compute a coarse description D of $\mathcal{E}(X)$. Since $\mathcal{E}$ is a uniform coarsening by Proposition 2.3, from D we can uniformly obtain sets $A_0, A_1, \dots$ with $A_s = X$ for all sufficiently large s. Composing these various reductions, from $X \setminus \{n\}$ we can uniformly compute sets $A_0, A_1, \dots$ with $A_s = X$ for all sufficiently large s. Then from $(X \setminus \{n\})'$ we can uniformly compute $\lim_s A_s(n) = X(n)$, as needed. $\square$

We will now show that 2-generic sets are not jump-autoreducible, which will give us a first example separating uniform coarse reducibility and nonuniform coarse reducibility. For this we first show that no 1-generic set is autoreducible, which is an easy exercise.

Proposition 2.6. *If X is 1-generic, then X is not autoreducible.*

Proof. Suppose for the sake of a contradiction that X is 1-generic and is autoreducible via Φ. For a string σ, let $\sigma^{-1}(i)$ be the set of n such that

$\sigma(n) = i$. If τ is a binary string, let $\tau \setminus \{n\}$ be the unique binary string μ of the same length such that $\mu^{-1}(1) = \tau^{-1}(1) \setminus \{n\}$. Let S be the set of strings τ such that $\Phi^{\tau\setminus\{n\}}(n)\downarrow \neq \tau(n)\downarrow$ for some n. Then S is a c.e. set of strings and X does not meet S. Since X is 1-generic, there is a string $\sigma \prec X$ that has no extension in S. Let $n = |\sigma|$, and let $\tau \succ \sigma$ be a string such that $\Phi^{\tau\setminus\{n\}}(n)\downarrow$. Such a string τ exists because $\sigma \prec X$ and Φ witnesses that X is autoreducible. Furthermore, we may assume that $\tau(n) \neq \Phi^{\tau\setminus\{n\}}$, since changing the value of $\tau(n)$ does not affect any of the conditions in the choice of τ. Hence τ is an extension of σ and $\tau \in S$, which is the desired contradiction. □

Proposition 2.7. *If X is 2-generic, then X is not jump-autoreducible.*

Proof. Since X is 2-generic, X is 1-generic relative to $\emptyset'$. Hence, by relativizing the proof of the previous proposition to $\emptyset'$, we see that X is not autoreducible relative to $\emptyset'$. However, the class of 1-generic sets is uniformly GL_1, i.e., there exists a single Turing functional Ψ such that for every 1-generic X we have $\Psi^{X\oplus\emptyset'} = X'$, as can be verified by looking at the usual proof that every 1-generic is GL_1 (see [12, Lemma 2.6]). Of course, if X is 1-generic, then $X \setminus \{n\}$ is also 1-generic for every n. Thus from an oracle for $(X \setminus \{n\}) \oplus \emptyset'$ we can uniformly compute $(X \setminus \{n\})'$. Now, if X is jump-autoreducible, from $(X \setminus \{n\})'$ we can uniformly compute $X(n)$. Composing these reductions shows that $X(n)$ is uniformly computable from $(X \setminus \{n\}) \oplus \emptyset'$, which contradicts our previous remark that X is not autoreducible relative to $\emptyset'$. □

Corollary 2.8. *If X is 2-generic, then $\mathcal{E}(X) \leqslant_{\mathrm{nc}} \mathcal{I}(X)$ but $\mathcal{E}(X) \nleqslant_{\mathrm{uc}} \mathcal{I}(X)$.*

Proof. We know that $\mathcal{E}(X) \leqslant_{\mathrm{nc}} \mathcal{I}(X)$ from Proposition 2.2. The fact that $\mathcal{E}(X) \nleqslant_{\mathrm{uc}} \mathcal{I}(X)$ follows from Propositions 2.5 and 2.7. □

It is natural to ask whether the same result holds for 2-random sets. In the proof above we used the fact that the 2-generic sets are uniformly GL_1. For 2-random sets this fact is almost true, as expressed by the following lemma. The proof is adapted from Monin [18], where a generalization for higher levels of randomness is proved. Let $U_0, U_1, \ldots$ be a fixed universal Martin-Löf test relative to $\emptyset'$. The 2-*randomness deficiency* of a 2-random X is the least c such that $X \notin U_c$.

Lemma 2.9. *There is a Turing functional Θ such that, for a 2-random X and an upper bound b on the 2-randomness deficiency of X, we have $\Theta^{X\oplus\emptyset',b} = X'$.*

Proof. Let $\mathcal{V}_e = \{Z : e \in Z'\}$. The $\mathcal{V}_e$ are uniformly Σ^0_1 classes, so we can define a function $f \leqslant_{\mathrm{T}} \emptyset'$ such that $\mu(\mathcal{V}_e \setminus \mathcal{V}_e[f(e,i)]) < 2^{-i}$ for all e and i. Then each sequence $\mathcal{V}_e \setminus \mathcal{V}_e[f(e,0)], \mathcal{V}_e \setminus \mathcal{V}_e[f(e,1)], \ldots$ is an $\emptyset'$-Martin Löf test, and from b we can compute a number m such that if X is 2-random and b bounds the 2-randomness deficiency of X, then $X \notin \mathcal{V}_e \setminus \mathcal{V}_e[f(e,m)]$. Then $X \in \mathcal{V}_e$ if and only if $X \in \mathcal{V}_e[f(e,m)]$, which we can verify $(X \oplus \emptyset')$-computably. □

Proposition 2.10. *If X is 2-random, then X is not jump-autoreducible.*

Proof. Because X is 2-random, it is not autoreducible relative to $\emptyset'$, as can be seen by relativizing the proof of Figueira, Miller, and Nies [8] that no 1-random set is autoreducible. To obtain a contradiction, assume that X is jump-autoreducible through some functional Φ. It can be directly verified that there is a computable function f such that $f(n)$ bounds the randomness deficiency of $X \setminus \{n\}$. Now let $\Psi^{Y \oplus \emptyset'}(n) = \Phi^{\Theta^{Y \oplus \emptyset', f(n)}}(n)$. Then X is autoreducible relative to $\emptyset'$ through Ψ, a contradiction. □

Corollary 2.11. *If X is 2-random, then $\mathcal{E}(X) \leqslant_{\mathrm{nc}} \mathcal{I}(X)$ but $\mathcal{E}(X) \nleqslant_{\mathrm{uc}} \mathcal{I}(X)$.*

Although we will not discuss generic reducibility after this section, it is worth noting that our maps $\mathcal{E}$ and $\mathcal{I}$ also allow us to distinguish generic reducibility from its nonuniform analog. Let us briefly review the relevant definitions from [13]. A *generic description* of a set A is a partial function that agrees with A where defined, and whose domain has density 1. A set A is *generically reducible* to a set B, written $A \leqslant_{\mathrm{g}} B$, if there is an enumeration operator W such that if Φ is a generic description of B, then $W^{\mathrm{graph}(\Phi)}$ is the graph of a generic description of A. We can define the notion of *nonuniform generic reducibility* in a similar way: $A \leqslant_{\mathrm{ng}} B$ if for every generic description Φ of B, there is a generic description Ψ of A such that graph(Ψ) is enumeration reducible to graph(Φ).

It is easy to see that $\mathcal{E}(X) \leqslant_{\mathrm{ng}} \mathcal{I}(X)$ for all X. On the other hand, we have the following fact.

Proposition 2.12. *If $\mathcal{E}(X) \leqslant_{\mathrm{g}} \mathcal{I}(X)$ then X is autoreducible.*

Proof. Let I_n be as in the definition of $\mathcal{I}$. Suppose that W witnesses that $\mathcal{E}(X) \leqslant_{\mathrm{g}} \mathcal{I}(X)$. We can assume that W^Z is the graph of a partial function for every oracle Z. Define a Turing functional Θ as follows. Given an oracle Y and an input n, let $\Phi(k) = Y(m)$ if $k \in I_m$ and $m \neq n$, and let $\Phi(k)\uparrow$ if $k \in I_n$. Let Ψ be the partial function with graph $W^{\mathrm{graph}(\Phi)}$. Search for an i and a $k \in I_{\langle n,i \rangle}$ such that $\Psi(k)\downarrow$. If such numbers are found then let $\Theta^Y(n) = \Psi(k)$. If $Y = X \setminus \{n\}$ then Φ is a generic description of $\mathcal{I}(X)$, so Ψ is a generic description of $\mathcal{E}(X)$, and hence $\Theta^Y(n)\downarrow = X(n)$. Thus X is autoreducible. □

We finish this section by showing that, for both the uniform and the nonuniform coarse degrees, coarsenings of the appropriate type preserve joins but do not always preserve existing meets.

Proposition 2.13. *Let $\mathcal{C}$ be a coarsening. Then $\mathcal{C}(A \oplus B)$ is the least upper bound of $\mathcal{C}(A)$ and $\mathcal{C}(B)$ in the nonuniform coarse degrees. The same holds for the uniform coarse degrees if $\mathcal{C}$ is a uniform coarsening.*

Proof. By Proposition 2.2 we know that $\mathcal{C}(A \oplus B)$ is an upper bound for $\mathcal{C}(A)$ and $\mathcal{C}(B)$ in both the uniform and nonuniform coarse degrees. Let us show that it is the least upper bound. If $\mathcal{C}(A), \mathcal{C}(B) \leqslant_{\mathrm{nc}} G$ then every coarse description D of G computes both A and B, so $D \geqslant_{\mathrm{T}} A \oplus B \geqslant_{\mathrm{T}} \mathcal{C}(A \oplus B)$. Thus $G \geqslant_{\mathrm{nc}} \mathcal{C}(A \oplus B)$.

Finally, assume that $\mathcal{C}$ is a uniform coarsening and let $\mathcal{C}(A), \mathcal{C}(B) \leqslant_{\mathrm{uc}} G$. Let Φ be a Turing functional such that $\Phi^{A \oplus B} = \mathcal{C}(A \oplus B)$. Every coarse description H of G uniformly computes coarse descriptions D_1 of $\mathcal{C}(A)$ and D_2 of $\mathcal{C}(B)$. Since $\mathcal{C}$ is uniform, there are Turing functionals Γ and Δ such that, letting $A_s(n) = \Gamma^{D_1}(n, s)$ and $B_s(n) = \Gamma^{D_2}(n, s)$, we have that $A \oplus B = A_s \oplus B_s$ for all sufficiently large s. Let E be defined as follows. Given n, search for an $s \geqslant n$ such that $\Phi^{A_s \oplus B_s}(n)\downarrow$, and let $E(n) = \Phi^{A_s \oplus B_s}(n)$. If n is sufficiently large, then $E(n) = \Phi^{A \oplus B}(n) = \mathcal{C}(A \oplus B)(n)$, so E is a coarse description of $\mathcal{C}(A \oplus B)$. Since E is obtained uniformly from H, we have that $\mathcal{C}(A \oplus B) \leqslant_{\mathrm{uc}} G$. □

Lemma 2.14. *Let $\mathcal{C}$ be a uniform coarsening and let $Y \leqslant_{\mathrm{T}} X$. Then $Y \leqslant_{\mathrm{uc}} \mathcal{C}(X)$.*

Proof. Let Φ be a Turing functional such that $\Phi^X = Y$. Let D be a coarse description of $\mathcal{C}(X)$ and let A_s be as in Definition 2.1. Now define $G(n)$ to be the value of $\Phi^{A_s}(n)$ for the least pair $\langle s, t \rangle$ such that $s \geqslant n$ and $\Phi^{A_s}(n)[t]\downarrow$. Then $G =^* Y$, so G is a coarse description of Y. □

Proposition 2.15. *Let $\mathcal{C}$ be a coarsening. Then $\mathcal{C}$ does not always preserve existing meets in the nonuniform coarse degrees. The same holds for the uniform coarse degrees if $\mathcal{C}$ is a uniform coarsening.*

Proof. Let X, Y be relatively 2-random and Δ^0_3. Then X and Y form a minimal pair in the Turing degrees, while X and Y do not form a minimal pair in the nonuniform coarse degrees by Theorem 5.6 below. Since every coarse description of $\mathcal{C}(X)$ computes X we see that $\mathcal{C}(X) \geqslant_{\mathrm{nc}} X$ and $\mathcal{C}(Y) \geqslant_{\mathrm{nc}} Y$. Therefore $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ also do not form a minimal pair in the nonuniform coarse degrees.

Next, let $\mathcal{C}$ be a uniform coarsening. We have seen above that there exists some $A \leqslant_{\mathrm{nc}} \mathcal{C}(X), \mathcal{C}(Y)$ that is not coarsely computable. Then $A \leqslant_{\mathrm{T}} X, Y$, so $A \leqslant_{\mathrm{uc}} \mathcal{C}(X), \mathcal{C}(Y)$ by the previous lemma. Thus, $\mathcal{C}(X)$ and $\mathcal{C}(Y)$ do not form a minimal pair in the uniform coarse degrees. □

3. Randomness, K-triviality, and robust information coding

It is reasonable to expect that the embeddings induced by $\mathcal{E}$ (or equivalently, by any uniform coarsening) are not surjective. Indeed, if $\mathcal{E}(A) \leqslant_{\mathrm{uc}}$

X then the information represented by A is coded into X in a fairly redundant way. If A is noncomputable, it should follow that X cannot be random. As we will see, we can make this intuition precise.

Definition 3.1. Let $X^{\mathfrak{c}}$ be the set of all A such that A is computable from every coarse description of X.

We will show that if X is weakly 2-random then $X^{\mathfrak{c}} = \mathbf{0}$, and hence $\mathcal{E}(A) \not\leqslant_{\mathrm{nc}} X$ for all noncomputable A (since every coarse description of $\mathcal{E}(A)$ computes A). Since no 1-random set can be coarsely computable, it will follow that $X \not\equiv_{\mathrm{nc}} \mathcal{E}(B)$ and $X \not\equiv_{\mathrm{uc}} \mathcal{E}(B)$ for all B. We will first prove the following theorem. Let $\mathcal{K}$ be the class of K-trivial sets. (See [3] or [19] for more on K-triviality.)

Theorem 3.2. *If X is 1-random then $X^{\mathfrak{c}} \subseteq \mathcal{K}$.*

By Downey, Nies, Weber, and Yu [6], if X is weakly 2-random then it cannot compute any noncomputable Δ^0_2 sets. Since $\mathcal{K} \subset \Delta^0_2$, our desired result follows from Theorem 3.2.

Corollary 3.3. *If X is weakly 2-random then $X^{\mathfrak{c}} = \mathbf{0}$, and hence $\mathcal{E}(A) \not\leqslant_{\mathrm{nc}} X$ for all noncomputable A. In particular, in both the uniform and nonuniform coarse degrees, the degree of X is not in the image of the embedding induced by $\mathcal{E}$.*

To prove Theorem 3.2, we use the fact, established by Hirschfeldt, Nies, and Stephan [11], that A is K-trivial if and only if A is a base for 1-randomness, that is, A is computable in a set that is 1-random relative to A. The basic idea is to show that if X is 1-random and $A \in X^{\mathfrak{c}}$, then for each $k > 1$ there is a way to partition X into k many "slices" $X_0, \dots, X_{k-1}$ such that for each $i < k$, we have $A \leqslant_{\mathrm{T}} X_0 \oplus \cdots \oplus X_{i-1} \oplus X_{i+1} \oplus \cdots \oplus X_{k-1}$ (where the right hand side of this inequality denotes $X_1 \oplus \cdots \oplus X_{k-1}$ when $i = 0$ and $X_0 \oplus \cdots \oplus X_{k-2}$ when $i = k-1$). It will then follow by van Lambalgen's Theorem (which will be discussed below) that each X_i is 1-random relative to $X_0 \oplus \cdots \oplus X_{i-1} \oplus X_{i+1} \oplus \cdots \oplus X_{k-1} \oplus A$, and hence, again by van Lambalgen's Theorem, that X is 1-random relative to A. Since $A \in X^{\mathfrak{c}}$ implies that $A \leqslant_{\mathrm{T}} X$, we will conclude that A is a base for 1-randomness, and hence is K-trivial. We begin with some notation for certain partitions of X.

Definition 3.4. Let $X \subseteq \omega$. For an infinite subset $Z = \{z_0 < z_1 < \cdots\}$ of ω, let $X \upharpoonright Z = \{n : z_n \in X\}$. For $k > 1$ and $i < k$, define

$$X^k_i = X \upharpoonright \{n : n \equiv i \bmod k\} \quad \text{and} \quad X^k_{\neq i} = X \upharpoonright \{n : n \not\equiv i \bmod k\}.$$

Note that $X^k_{\neq i} \equiv_{\mathrm{T}} X \setminus \{n : n \equiv i \bmod k\}$ and $\overline{\rho}(X \triangle (X \setminus \{n : n \equiv i \bmod k\})) \leqslant \frac{1}{k}$.

Van Lambalgen's Theorem [23] states that $Y \oplus Z$ is 1-random if and only if Y and Z are relatively 1-random. The proof of this theorem shows, more generally, that if Z is computable, infinite, and coinfinite, then X is 1-random if and only if $X \upharpoonright Z$ and $X \upharpoonright \overline{Z}$ are relatively 1-random. Relativizing this fact and applying induction, we get the following version of van Lambalgen's Theorem.

Theorem 3.5 (van Lambalgen [23])**.** *The following are equivalent for all sets X and A, and all $k > 1$.*

1. *X is 1-random relative to A.*
2. *For each $i < k$, the set X_i^k is 1-random relative to $X_{\neq i}^k \oplus A$.*

The last ingredient we need for the proof of Theorem 3.2 is a kind of compactness principle, which will also be used to yield further results in the next section, and is of independent interest given its connection with the following concept defined in [10].

Definition 3.6. Let $r \in [0,1]$. A set X is *coarsely computable at density r* if there is a computable set C such that $\overline{\rho}(X \triangle C) \leqslant 1-r$. The *coarse computability bound* of X is

$$\gamma(X) = \sup\{r : X \text{ is coarsely computable at density } r\}.$$

As noted in [10], there are sets X such that $\gamma(X) = 1$ but X is not coarsely computable. In other words, there is no principle of "compactness of computable coarse descriptions". (Although Miller (see [10, Theorem 5.8]) showed that one can in fact recover such a principle by adding a further effectivity condition to the requirement that $\gamma(X) = 1$.) The following theorem shows that if we replace "computable" by "cone-avoiding", the situation is different.

Theorem 3.7. *Let A and X be arbitrary sets. Suppose that for each $\varepsilon > 0$ there is a set D_ε such that $\overline{\rho}(X \triangle D_\varepsilon) \leqslant \varepsilon$ and $A \nleqslant_{\mathrm{T}} D_\varepsilon$. Then there is a coarse description D of X such that $A \nleqslant_{\mathrm{T}} D$.*

Proof. The basic idea is that, given a Turing functional Φ and a string σ that is "close to" X, we can extend σ to a string τ that is "close to" X such that $\Phi^D \neq A$ for all D extending τ that are "close to" X. We can take τ to be any string "close to" X such that, for some n, either $\Phi^\tau(n)\downarrow \neq A(n)$ or $\Phi^\gamma(n)\uparrow$ for all γ extending τ that are "close to" X. If no such τ exists, we can obtain a contradiction by arguing that $A \leqslant_{\mathrm{T}} D_\varepsilon$ for sufficiently small ε, since with an oracle for D_ε we have access to many strings that are "close to" D_ε and hence to X, by the triangle inequality for Hamming distance. In the above discussion the meaning of "close to" is different in different contexts, but the precise version will be given below. Further, as the construction proceeds, the meaning of "close to" becomes so stringent that we guarantee that $\rho(X \triangle D) = 0$. We now specify the formal details.

We obtain D as $\bigcup_e \sigma_e$, where $\sigma_e \in 2^{<\omega}$ and $\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots$. In order to ensure that $\rho(X \triangle D) = 0$, we require that for all e and all m in the interval $[|\sigma_e|, |\sigma_{e+1}|]$, either D and X agree on the interval $[|\sigma_e|, m)$ or $\rho_m(X \triangle D) \leqslant 2^{-|\sigma_e|}$, with the latter true for $m = |\sigma_{e+1}|$. This condition implies that $\rho_m(X \triangle D) \leqslant 2^{-|\sigma_e|}$ for all $m \in [|\sigma_{e+1}|, |\sigma_{e+2}|]$, and hence that $\rho(X \triangle D) = 0$.

Let σ and τ be strings and let ε be a positive real number. Call τ an *ε-good* extension of σ if τ properly extends σ and for all $m \in [|\sigma|, |\tau|]$, either X and τ agree on $[|\sigma|, m)$ or $\rho_m(\tau \triangle X) \leqslant \varepsilon$, with the latter true for $m = |\tau|$. In line with the previous paragraph, we require that σ_{e+1} be a $2^{-|\sigma_e|}$-good extension of σ_e for all e.

At stage 0, let σ_0 be the empty string. At stage $e+1$, we are given σ_e and choose σ_{e+1} as follows so as to force that $A \neq \Phi_e^D$. Let $\varepsilon = 2^{-|\sigma_e|}$.

Case 1. There is a number n and a string τ that is an ε-good extension of σ_e such that $\Phi_e^\tau(n)\downarrow \neq A(n)$. Let σ_{e+1} be such a τ.

Case 2. Case 1 does not hold and there is a number n and a string β that is an ε-good extension of σ_e such that $|\beta| \geqslant |\sigma_e| + 2$ and $\Phi_e^\tau(n)\uparrow$ for all $\frac{\varepsilon}{4}$-good extensions τ of β. Let σ_{e+1} be such a β.

We claim that either Case 1 or Case 2 applies. Suppose not. Let $D_{\frac{\varepsilon}{5}}$ be as in the hypothesis of the lemma, so that $\overline{\rho}(X \triangle D_{\frac{\varepsilon}{5}}) \leqslant \frac{\varepsilon}{5}$ and $A \nleqslant_{\mathrm{T}} D_{\frac{\varepsilon}{5}}$. Let $c \geqslant |\sigma_e| + 2$ be sufficiently large so that $\rho_m(X \triangle D_{\frac{\varepsilon}{5}}) \leqslant \frac{\varepsilon}{4}$ for all $m \geqslant c$ and σ_e has an $\frac{\varepsilon}{4}$-good extension β of length c. Note that the string obtained from σ_e by appending a sufficiently long segment of X starting with $X(|\sigma_e|)$ is an $\frac{\varepsilon}{4}$-good extension of σ_e, so such a β exists, and we assume it is obtained in this manner.

We now obtain a contradiction by showing that $A \leqslant_{\mathrm{T}} D_{\frac{\varepsilon}{5}}$. To calculate $A(n)$ search for a string γ extending β such that $\Phi_e^\gamma(n)\downarrow$, say with use u, and $\rho_m(D_{\frac{\varepsilon}{5}} \triangle \gamma) \leqslant \frac{\varepsilon}{2}$ for all $m \in [c, u)$. We first check that such a string γ exists. Since Case 2 does not hold, there is a string τ that is an $\frac{\varepsilon}{4}$-good extension of β such that $\Phi_e^\tau(n)\downarrow$. We claim that τ meets the criteria to serve as γ. We need only check that $\rho_m(D_{\frac{\varepsilon}{5}} \triangle \tau) \leqslant \frac{\varepsilon}{2}$ for all $m \in [c, u)$. Fix $m \in [c, u)$. Then

$$\rho_m(D_{\frac{\varepsilon}{5}} \triangle \tau) \leqslant \rho_m(D_{\frac{\varepsilon}{5}} \triangle X) + \rho_m(X \triangle \tau) \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$

Next we claim that γ is an ε-good extension of σ_e. The string γ extends σ_e since it extends β, and β extends σ_e. Let $m \in [|\sigma_e, |\gamma|]$ be given. If $m < c$, then γ and X agree on the interval $[|\sigma_e|, m)$ because β and X agree on this interval and γ extends β. Now suppose that $m \geqslant c$. Then

$$\rho_m(\gamma \triangle X) \leqslant \rho_m(\gamma \triangle D_{\frac{\varepsilon}{5}}) + \rho_m(D_{\frac{\varepsilon}{5}} \triangle X) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon.$$

Since γ is an ε-good extension of σ_e for which $\Phi_e^\gamma(n)\downarrow$, and Case 1 does not hold, we conclude that $\Phi_e^\gamma(n) = A(n)$. The search for γ can be carried out computably in $D_{\frac{\varepsilon}{5}}$, so we conclude that $A \leqslant_{\mathrm{T}} D_{\frac{\varepsilon}{5}}$, contradicting our choice of $D_{\frac{\varepsilon}{5}}$. (Although β cannot be computed from $D_{\frac{\varepsilon}{5}}$, we may use it in our computation of $A(n)$ since it is a fixed string which does not depend on n.) This contradiction shows that Case 1 or Case 2 must apply.

Let $D = \bigcup_n \sigma_n$. Then $\rho(D \triangle X) = 0$, and $A \nleqslant_{\mathrm{T}} D$ since Case 1 or Case 2 applies at every stage. □

Proof of Theorem 3.2. Let $A \in X^{\mathfrak{c}}$. By Theorem 3.7, there is an $\varepsilon > 0$ such that $A \leqslant_{\mathrm{T}} D_\varepsilon$ whenever $\overline{\rho}(X \triangle D_\varepsilon) \leqslant \varepsilon$. Let k be an integer such that $k > \frac{1}{\varepsilon}$. As noted in Definition 3.4, $X^k_{\neq i}$ is Turing equivalent to such a D_ε for each $i < k$, so we have $A \leqslant_{\mathrm{T}} X^k_{\neq i}$ for all $i < k$. By the unrelativized form of Theorem 3.5, each X^k_i is 1-random relative to $X^k_{\neq i}$, and hence relative to $X^k_{\neq i} \oplus A \equiv_{\mathrm{T}} X^k_{\neq i}$. Again by Theorem 3.5, X is 1-random relative to A. But $A \leqslant_{\mathrm{T}} X$, so A is a base for 1-randomness, and hence is K-trivial. □

Weak 2-randomness is exactly the level of randomness necessary to obtain Corollary 3.3 directly from Theorem 3.2, because, as shown in [6], if a 1-random set is not weakly 2-random, then it computes a noncomputable c.e. set. The corollary itself does hold of some 1-random sets that are not weakly 2-random, because if it holds of X then it also holds of any Y such that $\rho(Y \triangle X) = 0$. (For example, let X be 2-random and let Y be obtained from X by letting $Y(2^n) = \Omega(n)$ (where Ω is Chaitin's halting probability) for all n and letting $Y(k) = X(k)$ for all other k. By van Lambalgen's Theorem, Y is 1-random, but it computes Ω, and hence is not weakly 2-random.)

Nevertheless, Corollary 3.3 does not hold of all 1-random sets, as we now show.

Definition 3.8. Let $W_0, W_1, \ldots$ be an effective listing of the c.e. sets. A set A is *promptly simple* if it is c.e. and coinfinite, and there exist a computable function f and a computable enumeration $A[0], A[1], \ldots$ of A such that for each e, if W_e is infinite then there are n and s for which $n \in W_e[s] \setminus W_e[s-1]$ and $n \in A[f(s)]$. Note that every promptly simple set is noncomputable.

We will show that if $X \leqslant_{\mathrm{T}} \emptyset'$ is 1-random then $X^{\mathfrak{c}}$ contains a promptly simple set, and there is a promptly simple set A such that $\mathcal{E}(A) \leqslant_{\mathrm{nc}} X$. (We do not know whether we can improve the last statement to $\mathcal{E}(A) \leqslant_{\mathrm{uc}} X$.) In fact, we will obtain a considerably stronger result by first proving a generalization of the fact, due to Hirschfeldt and Miller (see [3, Theorem 7.2.11]), that if $\mathcal{T}$ is a Σ^0_3 class of measure 0, then there is a noncomputable c.e. set that is computable from each 1-random element of $\mathcal{T}$.

For a binary relation $P(Y, Z)$ between elements of 2^ω, let $P(Y) = \{Z : P(Y, Z)\}$.

Theorem 3.9. *Let $\mathcal{S}_0, \mathcal{S}_1, \ldots$ be uniformly Π^0_2 classes of measure 0, and let $P_0(Y, Z), P_1(Y, Z), \ldots$ be uniformly Π^0_1 relations. Let $\mathcal{D}$ be the class of* *all Y for which there are numbers k, m and a 1-random set Z such that $Z \in P_k(Y) \subseteq \mathcal{S}_m$. Then there is a promptly simple set A such that $A \leqslant_T Y$ for every $Y \in \mathcal{D}$.*

Proof. Let $(\mathcal{V}_n^m)_{m,n\in\omega}$ be uniformly Σ_1^0 classes such that $\mathcal{S}_m = \bigcap_n \mathcal{V}_n^m$. We may assume that $\mathcal{V}_0^m \supseteq \mathcal{V}_1^m \supseteq \cdots$ for all m. For each m, we have $\mu(\bigcap_n \mathcal{V}_m^n) = \mu(\mathcal{S}_m) = 0$, so $\lim_n \mu(\mathcal{V}_n^m) = 0$ for each m. Let Θ be a computable relation such that $P_k(Y, Z) \equiv \forall l\, \Theta(k, Y \upharpoonright l, Z \upharpoonright l)$.

Define A as follows. At each stage s, if there is an $e < s$ such that no numbers have entered A for the sake of e yet, and an $n > 2e$ such that $n \in W_e[s] \setminus W_e[s-1]$ and $\mu(\mathcal{V}_n^m[s]) \leqslant 2^{-e}$ for all $m < e$, then for the least such e, put the least corresponding n into A. We say that n enters A for the sake of e.

Clearly, A is c.e. and coinfinite, since at most e many numbers less than $2e$ ever enter A. Suppose that W_e is infinite. Let $t > e$ be a stage such that all numbers that will ever enter A for the sake of any $i < e$ are in $A[t]$. There must be an $s \geqslant t$ and an $n > 2e$ such that $n \in W_e[s] \setminus W_e[s-1]$ and $\mu(\mathcal{V}_n^m[s]) \leqslant 2^{-e}$ for all $m < e$. Then the least such n enters A for the sake of e at stage s unless another number has already entered A for the sake of e. It follows that A is promptly simple.

Now suppose that $Y \in \mathcal{D}$. Let the numbers k, m and the 1-random set Z be such that $Z \in P_k(Y) \subseteq \mathcal{S}_m$. Let $B \leqslant_T Y$ be defined as follows. Given n, let

$$\mathcal{D}_s^n = \{X : (\forall l \leqslant s)\, \Theta(k, Y \upharpoonright l, X \upharpoonright l)\} \setminus \mathcal{V}_n^m[s].$$

Then $\mathcal{D}_0^n \supseteq \mathcal{D}_1^n \supseteq \cdots$. Furthermore, if $X \in \bigcap_s \mathcal{D}_s^n$ then $P_k(Y, X)$ and $X \notin \mathcal{V}_n^m$. Since $P_k(Y) \subseteq S_m \subseteq \mathcal{V}_n^m$, it follows that $X \notin P_k(Y)$, which is a contradiction. Thus $\bigcap_s \mathcal{D}_s^n = \emptyset$. Since the $\mathcal{D}_s^n$ are nested closed sets, it follows that there is an s such that $\mathcal{D}_s^n = \emptyset$. Let s_n be the least such s (which we can find using Y) and let $B(n) = A(n)[s_n]$. Note that $B \subseteq A$.

Let $T = \{\mathcal{V}_n^m[s] : n \text{ enters } A \text{ at stage } s\}$. We can think of T as a uniform singly-indexed sequence of Σ_1^0 sets since m is fixed and for each n there is at most one s such that $\mathcal{V}_n^m[s] \in T$. For each e, there is at most one n that enters A for the sake of e, and the sum of the measures of the $\mathcal{V}_n^m[s]$ such that n enters A at stage s for the sake of some $e > m$ is bounded by $\sum_e 2^{-e}$, which is finite. Thus T is a Solovay test, and hence Z is in only finitely many elements of T. So for all but finitely many n, if n enters A at stage s then $Z \notin \mathcal{V}_n^m[s]$. Then $Z \in \mathcal{D}_s^n$, so $s_n > s$. Hence, for all such n, we have that $B(n) = A(n)[s_n] = 1$. Thus $B =^* A$, so $A \equiv_T B \leqslant_T Y$. $\square$

Note that the result of Hirschfeldt and Miller mentioned above follows from this theorem by starting with a Σ_3^0 class $\mathcal{S} = \bigcap_m \mathcal{S}_m$ of measure 0 and letting each P_k be the identity relation.

Corollary 3.10. *Let $X \leqslant_T \emptyset'$ be 1-random. There is a promptly simple set A such that if $\overline{\rho}(D \triangle X) < \frac{1}{4}$ then $A \leqslant_T D$. In particular, X^{c} contains* *a promptly simple set, and there is a promptly simple set A such that $\mathcal{E}(A) \leqslant_{\mathrm{nc}} X$.*

Proof. Say that sets Y and Z are *r-close from m on* if whenever $m < n$, the Hamming distance between $Y \upharpoonright n$ and $Z \upharpoonright n$ (i.e., the number of bits on which these two strings differ) is at most rn.

Let $\mathcal{S}_m$ be the class of all Z such that X and Z are $\frac{1}{2}$-close from m on. Since X is Δ^0_2, the $\mathcal{S}_m$ are uniformly Π^0_2 classes. Furthermore, if X and Z are $\frac{1}{2}$-close from m on for some m, then Z cannot be 1-random relative to X (by the same argument that shows that if C is 1-random then there must be infinitely many n such that $C \upharpoonright n$ has more 1's than 0's), so $\mu(\mathcal{S}_m) = 0$ for all m. Let $P_m(Y, Z)$ hold if and only if Y and Z are $\frac{1}{4}$-close from m on. The P_m are clearly uniformly Π^0_1 relations.

Thus the hypotheses of Theorem 3.9 are satisfied. Let A be as in that theorem. Suppose that $\overline{\rho}(D \triangle X) < \frac{1}{4}$. Then there is an m such that D and X are $\frac{1}{4}$-close from m on. If D and Z are $\frac{1}{4}$-close from m on, then by the triangle inequality for Hamming distance, X and Z are $\frac{1}{2}$-close from m on. Thus $X \in P_m(D) \subseteq \mathcal{S}_m$, so $A \leqslant_{\mathrm{T}} D$. □

After learning about Corollary 3.10, Nies [20] gave a different but closely connected proof of this result, which works even for X of positive effective Hausdorff dimension, as long as we sufficiently decrease the bound $\frac{1}{4}$. However, even for X of effective Hausdorff dimension 1 his bound is much worse, namely $\frac{1}{20}$.

Maass, Shore, and Stob [17, Corollary 1.6] showed that if A and B are promptly simple then there is a promptly simple set G such that $G \leqslant_{\mathrm{T}} A$ and $G \leqslant_{\mathrm{T}} B$. Thus we have the following extension of Kučera's result [15] that two Δ^0_2 1-random sets cannot form a minimal pair, which will also be useful below.

Corollary 3.11. *Let $X_0, X_1 \leqslant_{\mathrm{T}} \emptyset'$ be 1-random. There is a promptly simple set A such that if $\overline{\rho}(D \triangle X_i) < \frac{1}{4}$ for some $i \in \{0, 1\}$ then $A \leqslant_{\mathrm{T}} D$.*

It is easy to adapt the proof of Corollary 3.10 to give a direct proof of Corollary 3.11, and indeed of the fact that for any uniformly $\emptyset'$-computable family $X_0, X_1, \ldots$ of 1-random sets, there is a promptly simple set A such that if $\overline{\rho}(D \triangle X_i) < \frac{1}{4}$ for some i then $A \leqslant_{\mathrm{T}} D$. (We let $\mathcal{S}_{\langle i,m\rangle}$ be the class of all Z such that X_i and Z are $\frac{1}{2}$-close from m on, and the rest of the proof is essentially as before.)

Given the many (and often surprising) characterizations of K-triviality, it is natural to ask whether there is a converse to Theorem 3.2 stating that if A is K-trivial then $A \in X^{\mathfrak{c}}$ for some 1-random X. We now show that is not the case, using a recent result of Bienvenu, Greenberg, Kučera, Nies, and Turetsky [2]. There are many notions of randomness tests in the theory of algorithmic randomness. Some, like Martin-Löf tests, correspond to significant levels of algorithmic randomness, while other, less obviously natural ones have nevertheless become important tools in the development of this theory. Balanced tests belong to the latter class.

Definition 3.12. Let $\mathcal{W}_0, \mathcal{W}_1, \ldots \subseteq 2^\omega$ be an effective list of all Σ^0_1 classes. A *balanced test* is a sequence $(\mathcal{U}_n)_{n\in\omega}$ of Σ^0_1 classes such that there is a computable binary function f with the following properties.

1. $|\{s : f(n,s+1) \neq f(n,s)\}| \leqslant O(2^n)$,
2. $\forall n\, \mathcal{U}_n = \mathcal{W}_{\lim_s f(n,s)}$, and
3. $\forall n\, \forall s\, \mu(\mathcal{W}_{f(n,s)}) \leqslant 2^{-n}$.

For $\sigma \in 2^{<\omega}$ and $X \in 2^\omega$, we write σX for the element of 2^ω obtained by concatenating σ and X.

Theorem 3.13 (Bienvenu, Greenberg, Kučera, Nies, and Turetsky [2]). *There are a K-trivial set A and a balanced test $(\mathcal{U}_n)_{n\in\omega}$ such that if $A \leqslant_{\mathrm{T}} X$ then there is a string σ with $\sigma X \in \bigcap_n \mathcal{U}_n$.*

We will also use the following measure-theoretic fact.

Theorem 3.14 (Loomis and Whitney [16]). *Let $\mathcal{S} \subseteq 2^\omega$ be open, and let $k \in \omega$. For $i < k$, let $\pi_i(\mathcal{S}) = \{Y^k_{\neq i} : Y \in \mathcal{S}\}$. Then $\mu(\mathcal{S})^{k-1} \leqslant \mu(\pi_0(\mathcal{S})) \cdots \mu(\pi_{k-1}(\mathcal{S}))$.*

Our result will follow from the following lemma.

Lemma 3.15. *Let X be 1-random, let $k > 1$, and let $(\mathcal{U}_n)_{n\in\omega}$ be a balanced test. There is an $i < k$ such that $X^k_{\neq i} \notin \bigcap_n \mathcal{U}_n$.*

Proof. Assume for a contradiction that $X^k_{\neq i} \in \bigcap_n \mathcal{U}_n$ for all $i < k$. Let

$$\mathcal{S}_{n,s} = \{Y : \forall i < k\ (Y^k_{\neq i} \in \mathcal{U}_n[s])\}$$

and let $\mathcal{S}_n = \bigcup_s \mathcal{S}_{n,s}$. By Theorem 3.14, $\mu(\mathcal{S}_{n,s})^{k-1} \leqslant \mu(\mathcal{U}_n[s])^k$, so $\mu(\mathcal{S}_n) \leqslant O(2^n)2^{-\frac{nk}{k-1}} = O(2^{-\frac{n}{k-1}})$, and hence $\sum_n \mu(\mathcal{S}_n) < \infty$. Thus $\{\mathcal{S}_n : n \in \omega\}$ is a Solovay test. However, $X \in \bigcap_n \mathcal{S}_n$, so we have a contradiction. □

Theorem 3.16. *There is a K-trivial set A such that $A \notin X^{\mathfrak{c}}$ for all 1-random X.*

Proof. Let A and $(\mathcal{U}_n)_{n\in\omega}$ be as in Theorem 3.13. Let X be 1-random. By Theorem 3.7, it is enough to fix $k > 1$ and show that there is an $i < k$ such that $A \nleqslant_{\mathrm{T}} X^k_{\neq i}$. Assume for a contradiction that $A \leqslant_{\mathrm{T}} X^k_{\neq i}$ for all $i < k$. Then there are $\sigma_0, \ldots, \sigma_{k-1}$ such that $\sigma_i X^k_{\neq i} \in \bigcap_n \mathcal{U}_n$ for all $i < k$. Let $m = \max_{i<k} |\sigma_i|$ and let $\mathcal{V}_n = \{Y : \exists i < k\ (\sigma_i Y \in \mathcal{U}_{n+k+m})\}$. It is easy to check that $(\mathcal{V}_n)_{n\in\omega}$ is a balanced test, and $X^k_{\neq i} \in \bigcap_n \mathcal{V}_n$ for all $i < k$, which contradicts Lemma 3.15. □

4. Further applications of cone-avoiding compactness

We can use Theorem 3.7 to give an analog to Corollary 3.3 for effective genericity. In this case, 1-genericity is sufficient, as it is straightforward to show that if X is 1-generic relative to A and A is noncomputable, then $A \nleqslant_{\mathrm{T}} X$ (i.e., unlike the case for 1-randomness, there are no noncomputable bases for 1-genericity), and that no 1-generic set can be coarsely computable. The other ingredient we need to replicate the argument we gave in the case of effective randomness is a version of van Lambalgen's Theorem for 1-genericity. This result was established by Yu [24, Proposition 2.2]. Relativizing his theorem and applying induction as in the case of Theorem 3.5, we obtain the following fact.

Theorem 4.1 (Yu [24]). *The following are equivalent for all sets X and A, and all $k > 1$.*

1. *X is 1-generic relative to A.*
2. *For each $i < k$, the set X_i^k is 1-generic relative to $X_{\neq i}^k \oplus A$.*

Now we can establish the following analog to Corollary 3.3.

Theorem 4.2. *If X is 1-generic then $X^{\mathfrak{c}} = \mathbf{0}$, and hence $\mathcal{E}(A) \nleqslant_{\mathrm{nc}} X$ for all noncomputable A. In particular, in both the uniform and nonuniform coarse degrees, the degree of X is not in the image of the embedding induced by $\mathcal{E}$.*

Proof. Let $A \in X^{\mathfrak{c}}$. As in the proof of Theorem 3.2, there is a k such that $A \leqslant_{\mathrm{T}} X_{\neq i}^k$ for all $i < k$. By the unrelativized form of Theorem 4.1, each X_i^k is 1-generic relative to $X_{\neq i}^k$, and hence relative to $X_{\neq i}^k \oplus A \equiv_{\mathrm{T}} X_{\neq i}^k$. Again by Theorem 4.1, X is 1-generic relative to A. But $A \leqslant_{\mathrm{T}} X$, so A is computable. $\square$

Igusa (personal communication) has also found the following application of Theorem 3.7. We say that X is *generically computable* if there is a partial computable function φ such that $\varphi(n) = X(n)$ for all n in the domain of φ, and the domain of φ has density 1. Jockusch and Schupp [13, Theorem 2.26] showed that there are generically computable sets that are not coarsely computable, but by Lemma 1.7 in [10], if X is generically computable then $\gamma(X) = 1$, where γ is the coarse computability bound from Definition 3.6.

Theorem 4.3 (Igusa, personal communication). *If $\gamma(X) = 1$ then $X^{\mathfrak{c}} = \mathbf{0}$, and hence $\mathcal{E}(A) \nleqslant_{\mathrm{nc}} X$ for all noncomputable A. Thus, if $\gamma(X) = 1$ and X is not coarsely computable then in both the uniform and nonuniform coarse degrees, the degree of X is not in the image of the embedding induced by $\mathcal{E}$. In particular, the above holds when X is generically computable but not coarsely computable.*

Proof. Suppose that $\gamma(X) = 1$ and A is not computable. If $\varepsilon > 0$ then there is a computable set C such that $\overline{\rho}(X \triangle C) < \varepsilon$. Since C is computable, $A \nleqslant_{\mathrm{T}} C$. By Theorem 3.7, $A \notin X^{\mathfrak{c}}$. $\square$

5. Minimal pairs in the uniform and nonuniform coarse degrees

For any degree structure that acts as a measure of information content, it is reasonable to expect that if two sets are sufficiently random relative to each other, then their degrees form a minimal pair. For the Turing degrees, it is not difficult to show that if Y is not computable and X is weakly 2-random relative to Y, then the degrees of X and Y form a minimal pair. On the other hand, Kučera [15] showed that if $X, Y \leqslant_{\mathrm{T}} \emptyset'$ are both 1-random, then there is a noncomputable set $A \leqslant_{\mathrm{T}} X, Y$, so there are relatively 1-random sets whose degrees do not form a minimal pair. As we will see, the situation for the nonuniform coarse degrees is similar, but "one jump up".

For an interval I, let $\rho_I(X) = \frac{|X \cap I|}{|I|}$.

Lemma 5.1. *Let* $J_k = [2^k - 1, 2^{k+1} - 1)$*. Then* $\rho(X) = 0$ *if and only if* $\lim_k \rho_{J_k}(X) = 0$.

Proof. First suppose that $\limsup_k \rho_{J_k}(X) > 0$. Since $|J_k| = 2^k$, we have $\overline{\rho}(X) \geqslant \limsup_k \rho_{2^{k+1}-1}(X) \geqslant \limsup_k \frac{\rho_{J_k}(X)}{2} > 0$.

Now suppose that $\limsup_k \rho_{J_k}(X) = 0$. Fix $\varepsilon > 0$. If m is sufficiently large, $k \geqslant m$, and $n \in J_k$, then

$$|X \cap [0, n)| \leqslant |X \cap [0, 2^{k+1} - 1)| \leqslant \sum_{i=0}^{m-1} |J_i| + \sum_{i=m}^{k} \frac{\varepsilon}{2} |J_i|.$$

If k is sufficiently large then this sum is less than $\varepsilon(2^k - 1)$, whence $\rho_n(X) < \frac{\varepsilon(2^k-1)}{n} \leqslant \frac{\varepsilon n}{n} = \varepsilon$. Thus $\limsup_n \rho_n(X) \leqslant \varepsilon$. Since ε is arbitrary, $\limsup_n \rho_n(X) = 0$. $\square$

Theorem 5.2. *If A is not coarsely computable and X is weakly* 3*-random relative to A, then there is no X-computable coarse description of A. In particular,* $A \nleqslant_{\mathrm{nc}} X$.

Proof. Suppose that Φ^X is a coarse description of A and let

$$\mathcal{P} = \{Y : \Phi^Y \text{ is a coarse description of } A\}.$$

Then $Y \in \mathcal{P}$ if and only if

1. Φ^Y is total, which is a Π^0_2 property, and
2. for each k there is an m such that, for all $n > m$, we have $\rho_n(\Phi^Y \triangle A) < 2^{-k}$, which is a $\Pi^{0,A}_3$ property.

Thus $\mathcal{P}$ is a $\Pi^{0,A}_3$ class, so it suffices to show that if A is not coarsely computable then $\mu(\mathcal{P}) = 0$.

We prove the contrapositive. Suppose that $\mu(\mathcal{P}) > 0$. Then, by the Lebesgue Density Theorem, there is a σ such that $\mu(\mathcal{P} \cap [\![\sigma]\!]) > \frac{3}{4}2^{-|\sigma|}$. It is now easy to define a Turing functional Ψ such that the measure of the class of Y for which Ψ^Y is a coarse description of A is greater than $\frac{3}{4}$. Define a computable set D as follows. Let $J_k = [2^k - 1, 2^{k+1} - 1)$. For each k, wait until we find a finite set of strings S_k such that $\mu([\![S_k]\!]) > \frac{3}{4}$ and Ψ^σ converges on all of J_k for each $\sigma \in S_k$ (which must happen, by our choice of Ψ). Let n_k be largest such that there is a set $R_k \subseteq S_k$ with $\mu([\![R_k]\!]) > \frac{1}{2}$ and $\rho_{J_k}(\Psi^\sigma \triangle \Psi^\tau) \leqslant 2^{-n_k}$ for all $\sigma, \tau \in R_k$. Let $\sigma \in R_k$ and define $D \upharpoonright J_k = \Psi^\sigma \upharpoonright J_k$.

We claim that D is a coarse description of A. By Lemma 5.1, it is enough to show that $\lim_k \rho_{J_k}(D \triangle A) = 0$. Fix n. Let $\mathcal{B}_k$ be the class of all Y such that Ψ^Y converges on all of J_k and $\rho_{J_k}(\Psi^Y \triangle A) \leqslant 2^{-n}$. If Ψ^Y is a coarse description of A then, again by Lemma 5.1, $\rho_{J_k}(\Psi^Y \triangle A) \leqslant 2^{-n}$ for all sufficiently large k, so there is an m such that $\mu(\mathcal{B}_k) > \frac{3}{4}$ for each $k > m$, and hence $\mu(\mathcal{B}_k \cap [\![S_k]\!]) > \frac{1}{2}$ for each $k > m$. Let $T_k = \{\sigma \in S_k : \rho_{J_k}(\Psi^\sigma \triangle A) \leqslant 2^{-n}\}$. Then $[\![T_k]\!] = \mathcal{B}_k \cap [\![S_k]\!]$, so $\mu([\![T_k]\!]) > \frac{1}{2}$ for each $k > m$. Furthermore, by the triangle inequality for Hamming distance, $\rho_{J_k}(\Psi^\sigma \triangle \Psi^\tau) \leqslant 2^{-(n-1)}$ for all $\sigma, \tau \in T_k$. It follows that, for each $k > m$, we have $n_k \geqslant n - 1$, and at least one element Y of $\mathcal{B}_k$ is in $[\![R_k]\!]$ (where R_k is as in the definition of D), which implies that

$$\rho_{J_k}(D \triangle A) \leqslant \rho_{J_k}(D \triangle \Psi^Y) + \rho_{J_k}(\Psi^Y \triangle A) \leqslant 2^{-n_k} + 2^{-n} < 2^{-n+2}.$$

Since n is arbitrary, $\lim_k \rho_{J_k}(D \triangle A) = 0$. □

Corollary 5.3. *If Y is not coarsely computable and X is weakly 3-random relative to Y, then the nonuniform coarse degrees of X and Y form a minimal pair, and hence so do their uniform coarse degrees.*

Proof. Let $A \leqslant_{\mathrm{nc}} X, Y$. Then Y computes a coarse description D of A. We have $D \leqslant_{\mathrm{nc}} X$, and X is weakly 3-random relative to D, so by the theorem, D is coarsely computable, and hence so is A. □

For the nonuniform coarse degrees at least, this corollary does not hold of 2-randomness in place of weak 3-randomness. To establish this fact, we use the following complementary results. The first was proved by Downey, Jockusch, and Schupp [4, Corollary 3.16] in unrelativized form, but it is easy to check that their proof relativizes.

Theorem 5.4 (Downey, Jockusch, and Schupp [4])**.** *If A is c.e., $\rho(A)$ is defined, and $A' \leqslant_{\mathrm{T}} D'$, then D computes a coarse description of A.*

Theorem 5.5 (Hirschfeldt, Jockusch, McNicholl, and Schupp [10])**.** *Every nonlow c.e. degree contains a c.e. set A such that $\rho(A) = \frac{1}{2}$ and A is not coarsely computable.*

Theorem 5.6. *Let $X, Y \leqslant_{\mathrm{T}} \emptyset''$ (which is equivalent to $\mathcal{E}(X), \mathcal{E}(Y) \leqslant_{\mathrm{nc}} \mathcal{E}(\emptyset'')$). If X and Y are both 2-random, then there is an $A \leqslant_{\mathrm{nc}} X, Y$ such that A is not coarsely computable. In particular, there is a pair of relatively 2-random sets whose nonuniform coarse degrees do not form a minimal pair.*

Proof. Since X and Y are both 1-random relative to $\emptyset'$, by the relativized form of Corollary 3.11 there is an $\emptyset'$-c.e. set $J >_{\mathrm{T}} \emptyset'$ such that for every coarse description D of either X or Y, we have that $D \oplus \emptyset'$ computes J, and hence so does D'. By the Sacks Jump Inversion Theorem [21], there is a c.e. set B such that $B' \equiv_{\mathrm{T}} J$. By Theorem 5.5, there is a c.e. set $A \equiv_{\mathrm{T}} B$ such that $\rho(A) = \frac{1}{2}$ and A is not coarsely computable. Let D be a coarse description of either X or Y. Then $D' \geqslant_{\mathrm{T}} J \equiv_{\mathrm{T}} A'$, so by Theorem 5.4, D computes a coarse description of A. ☐

We do not know whether this theorem holds for uniform coarse reducibility.

6. Open Questions

We finish with a few questions raised by our results.

Open Question 6.1. Can the bound $\frac{1}{4}$ in Corollary 3.10 be increased?

Open Question 6.2. Let $X \leqslant_{\mathrm{T}} \emptyset'$ be 1-random. Must there be a noncomputable (c.e.) set A such that $\mathcal{E}(A) \leqslant_{\mathrm{uc}} X$? (Recall that Corollary 3.10 gives a positive answer to the nonuniform analog to this question.) If not, then is there any 1-random X for which such an A exists?

Open Question 6.3. Does Theorem 5.6 hold for uniform coarse reducibility?

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