NULLIFYING RANDOMNESS AND GENERICITY USING SYMMETRIC DIFFERENCE

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Abstract. For a class $\mathcal{C}$ of sets, let us say that a set A is $\mathcal{C}$ *stabilising* if $A \triangle X \in \mathcal{C}$ for every $X \in \mathcal{C}$. We prove that the Martin-Löf stabilising sets are exactly the K-trivial sets, as are the weakly 2-random stabilising sets. We also show that the 1-generic stabilising sets are exactly the computable sets.

1. Introduction

Lowness for randomness is a topic that has attracted much attention in the literature. Recall that a set A is *low for Martin-Löf randomness* if, whenever X is Martin-Löf random, X is also Martin-Löf random relative to A. Thus, if A is not low for Martin-Löf randomness, there is a Martin-Löf random set X such that A "derandomises" it.

One particular, very simple way in which an oracle A might derandomise a set X is if the symmetric difference $X \triangle A$ of X with A is not itself random. Here $X \triangle A = (X \setminus A) \cup (A \setminus X)$; or equivalently, if we identify sets with their indicator function, symmetric difference is the same as bitwise addition modulo 2. At first sight, it might seem that this method of derandomising a set is too weak to capture exactly those A that are not low for Martin-Löf randomness: it is very uniform, and also very local. Furthermore, it is a priori not even clear that the class of such A is degree-invariant, or that it is countable. However, at least the locality does not have to be a problem, since Nies [9] has shown that the (global) property of being low for Martin-Löf randomness corresponds to the (much more localised) property of being K-trivial.

Note that 2^ω forms an (abelian) group under the operation $\triangle$. In particular, we can view $\triangle$ as a group action $2^\omega \times 2^\omega \to 2^\omega$. Recall that, for any group action $G \times X \to X$, the *set stabiliser* of a subset $Y \subseteq X$ is the set

$$\{g \in G \mid gY = Y\}.$$

In case G is a torsion group (i.e., all elements have finite order), like 2^ω, note that

$$\{g \in G \mid gY = Y\} = \{g \in G \mid gY \subseteq Y\}.$$

Following this terminology, let us therefore make the following definition.

Definition 1.1. Let $\mathcal{C} \subseteq 2^\omega$. The *stabiliser* of $\mathcal{C}$ is the set

$$\{A \in 2^\omega \mid \forall X \in \mathcal{C}(A \triangle X \in \mathcal{C})\}.$$

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We will say that such an A is $\mathcal{C}$ *stabilising.*

The question of which sets are Martin-Löf stabilising has circulated in the effective randomness community (using various terminology). Kjos-Hanssen was probably the first person to ask it. The first time the question appeared in print seems to be in Kihara and Miyabe [4]. They study the stabiliser of various notions from randomness and genericity, mainly for its connection to the cardinal characteristic of null-additivity.

Kihara and Miyabe gave characterisations of the $\mathcal{C}$ stabiliser for several classes $\mathcal{C}$. Recall that, for a randomness notion $\mathcal{R}$, a set A is *uniformly low for $\mathcal{R}$-randomness* if every $\mathcal{R}$-random set X passes $\mathcal{T}^A$ for all $\mathcal{T}$ such that $\mathcal{T}^B$ is an $\mathcal{R}$-test for *every* oracle B (Miyabe [7] and Miyabe and Rute [8]). Recall that, for Martin-Löf randomness and 1-genericity, uniform lowness and lowness coincide. On the other hand, for many other notions this is not the case. Furthermore, as noted above, the map $X \mapsto A \triangle X$ is uniform in A, and therefore one would expect that a connection between $\mathcal{R}$ stabilising and lowness for $\mathcal{R}$-randomness would, if it exists at all, refer to uniform lowness instead of non-uniform lowness.

It turns out such a connection often exists. In [4], it is shown that the Kurtz stabilising sets are exactly the sets that are uniformly low for Kurtz randomness, and that the weakly 1-generic stabilising sets are those that are uniformly low for weak 1-genericity. Furthermore, if we generalise the notion of $\mathcal{C}$ stabilising to the notion of $(\mathcal{C}, \mathcal{D})$ stabilising, where A is $(\mathcal{C}, \mathcal{D})$ *stabilising* if for every $X \in \mathcal{C}$ we have that $A \triangle X \in \mathcal{D}$, then they have shown that (Martin-Löf, Schnorr) stabilising coincides with uniformly (Martin-Löf, Schnorr) low, and that (Martin-Löf, Kurtz) stabilising coincides with uniformly (Martin-Löf, Kurtz) low. Here, recall that A is $(\mathcal{C}, \mathcal{D})$ *low* if every $\mathcal{C}$-random is $\mathcal{D}$-random relative to A, and it is straightforward to formulate the uniform version of this.

We add several results to this list. We show that the Martin-Löf stabilising sets are those that are low for Martin-Löf randomness (i.e., the K-trivial sets), and that the 1-generic stabilising sets are exactly the sets that are low for 1-generic (i.e., computable, by Greenberg, Miller and Yu, as published in Yu [11]). We also show that the weakly 2-random stabilising sets are the sets that are low for weak 2-randomness, and that the sets that are (weakly 2-random, Martin-Löf) stabilising are those that are (weakly 2-random, Martin-Löf) low. Note that in these last two lowness classes also coincide with the K-trivial sets.

As noted above, these new characterisations of K-triviality are not obviously degree invariant. This is somewhat unusual. Of the seemingly countless characterisations of K-triviality that have been found, almost all of them say that K-trivial sets are weak as oracles, or that they are easy to compute (or both), in other words, properties that are explicitly degree invariant. The most notable exception to this rule, of course, is K-triviality itself.

The observant reader might have noticed that in all of the cases mentioned above, $\mathcal{C}$ stabilising coincides with being uniformly low for $\mathcal{C}$. Strangely enough, we do not know of a single meta-result that yields all of these. Instead, all of the proofs proceed indirectly; they each prove the equivalence of (uniform) lowness and stabilising by passing through a third characterisation. For example, we will show directly that Martin-Löf stabilising implies K-triviality, instead of reasoning directly about lowness for Martin-Löf randomness.

2. ML-randomness

In this section, we will prove that the sets that are Martin-Löf stabilising are exactly the K-trivial sets. Before we do so, we need to prove two easy lemmas.

Lemma 2.1. *For every universal Martin-Löf test $U_0, U_1, \ldots$, if A is Martin-Löf stabilising, then there is a Π^0_1-class P of positive measure and an $m \in \omega$ such that $A \triangle P \subseteq Q_m$, where Q_m is the complement of U_m.*

Proof. Towards a contradiction, let us assume the lemma is false Let P be a nonempty Π^0_1-class containing only Martin-Löf random sets. We will construct a set $X \in P$ for which $A \triangle X$ is not Martin-Löf random by a finite extension argument. That is, we will construct $\sigma_0 \preceq \sigma_1 \preceq \cdots$ such that $(A \triangle (P \cap [\![\sigma_i]\!])) \cap Q_i = \emptyset$ and such that $P \cap [\![\sigma_i]\!] \neq \emptyset$; then $X = \bigcup_{s\in\omega} \sigma_s$ is as desired.

We let $\sigma_{-1} = \emptyset$. Now, to define σ_s, since $P \cap [\![\sigma_{s-1}]\!]$ is nonempty and only contains Martin-Löf random sets, it has positive measure. Hence since we assumed the lemmas is false, there exists a string $\sigma \succeq \sigma_{s-1}$ with $[\![\sigma]\!] \cap P \neq \emptyset$ and $(A \triangle (P \cap [\![\sigma]\!])) \cap Q_s = \emptyset$. Now let $\sigma_s = \sigma$ for the least such string σ. ☐

The converse is also true, although we do not need it below.

Lemma 2.2. *If there is a Π^0_1-class P of positive measure and a Π^0_1-class Q containing only Martin-Löf random sets such that $A \triangle P \subseteq Q$, then A is Martin-Löf stabilising.*

Proof. Let X be Martin-Löf random. By Bievenu et al. [2, Theorem 2], there is a $Y =^* X$ such that $Y \in P$. Thus, $Y \triangle A$ is Martin-Löf random, and hence so is $X \triangle A =^* Y \triangle A$. ☐

Theorem 2.3. *A set A is Martin-Löf stabilising if and only if A is K-trivial.*

Proof. If A is K-trivial, then it is low for Martin-Löf randomness by Nies [9], and therefore it is Martin-Löf stabilising.

For the converse, let us first fix a universal Martin-Löf test $U_0 \supseteq U_1 \supseteq \cdots$ such that, for every Σ^0_1-class W_e and for every $i \in \omega$, if $\mu(W_e) \leq 2^{-e-i-2}$, then $W_e \subseteq U_i$. Let Q_i denote the complement of U_i.

Let B be Martin-Löf stabilising. By Lemma 2.1, there are a Π^0_1-class P of positive measure and an $m \in \omega$ such that $B \triangle P \subseteq Q_m$. Thus, it suffices to show that for every A with $A \triangle P \subseteq Q_m$ we have that A is K-trivial.

We are going to build a Π^0_1-class R and a request set L. By the recursion theorem, we may assume we know an index e for R. Hence, if we construct R in such a way that $\mu(R) \geq 1 - 2^{-e-m-2}$, then $Q_m \subseteq R$. This is what we will do, and therefore it suffices to show that for every A with $A \triangle P \subseteq R$ we have that A is K-trivial.

The idea is that we take blocks out of R in a highly independent way, and therefore, if A is indeed such that $A \triangle P \subseteq R$, this will force reals out of P, and these sets will be independent for different A. Then, we use the measure given to us by P to enumerate requests for the initial segments of A which will ensure that A is K-trivial.

Construction. First, let $f(\langle n, k\rangle)$ be the computable function recursively defined by $f(-1) = 0$ and $f(\langle n, k\rangle) = f(\langle n, k\rangle - 1) + 2^n(k + e + m + 2)$.

Next, let us define R. Let $R_0 = 2^\omega$. At stage s, we act for every number n such that $K_s(n) < K_{s-1}(n)$. For such an n, let $k = K_s(n)$. Let $\sigma_0, \ldots, \sigma_{2^n-1}$ list the strings of length n in lexicographic order. Now remove from R_s the set $V_{\langle n,k\rangle}$ consisting of all X for which, if $X \succeq \sigma_i$, then X restricted to the half open interval

$$[f(\langle n,k\rangle - 1) + (k+e+m+2)i, f(\langle n,k\rangle - 1) + (k+e+m+2)(i+1))$$

is $0^{k+e+m+2}$.

Finally, let us define L. For all σ of length at most s for which, if we let $n = |\sigma|$ and $k = K_s(n)$, there is a string $\tau \succeq \sigma$ of length $f(\langle n,k\rangle)$ such that $V_{\langle n,k\rangle} \subseteq \tau \triangle \overline{P_s}$, we enumerate a request (k,σ) into L.

Verification. First, let us show that R has measure at least $1 - 2^{-e-m-2}$. By our construction, we have that

$$\mu(R) \geq 1 - \sum_{n\in\omega} \sum_{\tau:\mathcal{U}(\tau)=n} 2^{-|\tau|-e-m-2} \geq 1 - 2^{-e-m-2}\Omega \geq 1 - 2^{-e-m-2},$$

where Ω is the halting probability of our universal machine $\mathcal{U}$.

Next, let us show that L is a request set, i.e., that $\sum_{(k,\sigma)\in L} 2^{-k} < \infty$. If we put (k,σ) into L, say at stage s, this means we found some $\tau_{k,\sigma} \succeq \sigma$ of length $f(\langle n,k\rangle)$ such that $V_{n,k} \subseteq \tau_{k,\sigma} \triangle \overline{P_s}$. Thus, $P_s \subseteq \overline{V_{\langle n,k\rangle}} \triangle \tau_{k,\sigma}$.

We claim that

$$\mu\left(\bigcap_{(k,\sigma)\in L} \overline{V_{\langle n,k\rangle}} \triangle \tau_{k,\sigma}\right) = \prod_{(k,\sigma)\in L} \mu\left(\overline{V_{\langle n,k\rangle}} \triangle \tau_{k,\sigma}\right).$$

Indeed, it is not hard to check that $X \in \bigcap_{(k,\sigma)\in L} \overline{V_{\langle n,k\rangle}} \triangle \tau_{k,\sigma}$ if and only if, for every $(k,\sigma) \in L$, if we let $n = |\sigma|$, then X satisfies the requirement that, if we let i be the position of $\sigma \triangle (X \upharpoonright n)$ in the lexicographic ordering of 2^n, then $X(j) \neq \tau_{k,\sigma}(j)$ for some

$$(1) \quad j \in [f(\langle n,k\rangle - 1) + (k+e+m+2)i, f(\langle n,k\rangle - 1) + (k+e+m+2)(i+1)).$$

Thus, if we have distinct $(k,\sigma), (k',\sigma') \in L$, there are three cases. If $|\sigma| \neq |\sigma'|$ or $k \neq k'$, then the requirements for (k,σ) and (k',σ') clearly act on different levels of the form (1). Otherwise, we have $|\sigma| = |\sigma'|$ and $k = k'$ but $\sigma \neq \sigma'$. Then for every X, we have that $\sigma \triangle (X \upharpoonright n)$ and $\sigma' \triangle (X \upharpoonright n)$ differ, so their position in the lexicographic ordering differ and hence their intervals in (1) are again disjoint.

So, we have that

$$\mu(P) = \mu\left(\bigcap_{s\in\omega} P_s\right) \leq \prod_{(k,\sigma)\in L} \mu(\overline{V_{\langle n,k\rangle}} \triangle \tau_{k,\sigma}) = \prod_{(k,\sigma)\in L} (1 - 2^{-k-e-m-2}).$$

In particular, $\prod_{(k,\sigma)\in L}(1 - 2^{-k-e-m-2}) > 0$ since P has positive measure. Thus, we know that $\sum_{(k,\sigma)\in L} 2^{-k-e-m-2} < \infty$, and hence also $\sum_{(k,\sigma)\in L} 2^{-k} < \infty$ (see e.g. Knopp [6, Chapter VII, Theorem 4]). This proves that L is a request set, so by the KC theorem we know that there is a constant c such that $K(\sigma) \leq k + c$ for every $(\sigma, k) \in L$.

Finally, assume that $A \triangle P \subseteq R$. Fix $n \in \omega$. If we let $k = K(n)$, we let t be the stage at which we act for $\langle n,k\rangle$, and we let $\tau = A \upharpoonright f(\langle n,k\rangle)$, then for large enough stages s we have $P_s \triangle \tau \subseteq R_t$. Therefore, we enumerate a request $(A \upharpoonright n, k)$ into L which implies that $K(A \upharpoonright n) \leq K(n) + c$. □

3. Weak 2-randomness

In this section we prove, using a slight modification of the argument given in the previous section, that the weakly 2-random stabilising sets and the (weakly 2-random, Martin-Löf) stabilising sets are also the K-trivial sets. For this, we show that the following weakened form of Lemma 2.1 still holds, by a slight modification of the proof given above.

Lemma 3.1. *For every universal Martin-Löf test $U_0, U_1, \dots$, if A is (weakly 2-random, Martin-Löf) stabilising, then there is a Π^0_1-class P of positive measure and an $m \in \omega$ such that $A \triangle P \subseteq Q_m$, where Q_m is the complement of U_m.*

Proof. Towards a contradiction, let us assume that the lemma is false. Let P_{-1} be a nonempty Π^0_1-class containing only Martin-Löf random sets. We will construct a sequence $P_{-1} \supseteq P_0 \supseteq \cdots$ of Π^0_1-classes of positive measure and a weakly 2-random set $X \in \bigcap_{i\in\omega} P_i$ for which $A \triangle X$ is not Martin-Löf random. We will approximate X by a finite initial segments, i.e., $X = \bigcup_{s\in\omega} \sigma_s$ for $\sigma_0 \preceq \sigma_1 \preceq \cdots$, where in addition we require that $P_i \cap [\![\sigma_i]\!] \neq \emptyset$. (So strictly speaking, we are forcing with Π^0_1-classes.)

At even stages $2s$, we let $P_{2s} = P_{2s-1}$ and we ensure that $[\![\sigma_{2s}]\!] \cap P_{2s} \neq \emptyset$ and $(A \triangle (P_{2s} \cap [\![\sigma_{2s}]\!])) \cap Q_s = \emptyset$, as in the proof of Lemma 2.1.

At odd stages $2s+1$, we ensure that X is weakly 2-random. We let $\sigma_{2s+1} = \sigma_{2s}$. Let $U = \bigcup_{i\in\omega} V_i$ be the Σ^0_2-class with index s. If U does not have measure 1, let $P_{2s+1} = P_{2s}$. Otherwise, note that $P_{2s} \cap [\![\sigma_{2s}]\!]$ has positive measure, since it is a nonempty Π^0_1-class of weakly 1-random sets. Let i be least such that $V_i \cap P_{2s} \cap [\![\sigma_{2s}]\!]$ has positive measure, and let $P_{2s+1} = V_i \cap P$.

Then the even stages ensure that $A \triangle X$ is not Martin-Löf random, while the odd stages ensure that X is contained in every Σ^0_2-set of measure 1, i.e., that X is weakly 2-random. □

Theorem 3.2. *Let $A \in 2^\omega$. The following are equivalent:*

1. *A is weakly 2-random stabilising.*
2. *A is (weakly 2-random, Martin-Löf) stabilising.*
3. *A is K-trivial.*

Proof. That (1) implies (2) is trivial. The proof that (2) implies (3) is exactly the same as for Theorem 2.3, except that we use Lemma 3.1 instead of Lemma 2.1. Finally, that (3) implies (1) follows from the fact that the K-trivial sets are low for weak 2-randomness, as shown independently by Nies [10] and Kjos-Hanssen, Miller and Solomon [5]. □

4. 1-Genericity

In this section, we prove that the sets that are 1-generic stabilising are exactly the computable sets. Perhaps surprisingly, this proof uses notions normally associated with randomness and not with genericity, such as Kolmogorov complexity and K-triviality. We first show that every 1-generic stabilising set is infinitely often K-trivial.

Definition 4.1 (Barmpalias and Vlek [1])**.** A set A is *infinitely often K-trivial* if there is a constant c such that $K(A \upharpoonright n) \leq K(n) + c$ for infinitely many n.

Theorem 4.2 (Barmpalias and Vlek [1])**.** *Every (weakly) 1-generic set is infinitely often K-trivial.*

Theorem 4.3. *If A is 1-generic stabilising, then it is infinitely often K-trivial.*

Proof. Muchnik proved that there is a noncomputable c.e. set C that is low for K, meaning that $(\forall\sigma)\ K(\sigma) \leq K^C(\sigma) + O(1)$. (Note that by Nies [9], low for K is yet another characterisation of K-triviality.) Since every noncomputable c.e. set computes a 1-generic set, we know that there is a 1-generic set $X \leq_T C$. Of course, X must also be low for K.

Now note that for all $n \in \omega$ we have

$$\begin{aligned} K(A \restriction n) \leq K^X(A \restriction n) + O(1) &\leq K^X((X \triangle A) \restriction n) + O(1) \\ &\leq K((X \triangle A) \restriction n) + O(1). \end{aligned}$$

So if we apply Theorem 4.2 to $X \triangle A$, we see that $K(A \restriction n) \leq K(n) + O(1)$ for infinitely many n, as desired. □

We now characterise the 1-generic stabilising sets.

Theorem 4.4. *A set A is 1-generic stabilising if and only if it is computable.*

Proof. First, note that every computable set is clearly 1-generic stabilising.

Conversely, let A be 1-generic stabilising. Towards a contradiction, assume that A is not computable. By Theorem 4.3, we know that A is infinitely often K-trivial.

We will construct an X such that X is 1-generic but $X \triangle A$ computes $\emptyset'$, which clearly implies that $X \triangle A$ is not 1-generic and hence A is not 1-generic stabilising, a contradiction. To make $X \triangle A$ compute $\emptyset'$, we code $\emptyset'$ into $X \triangle A$ while simultaneously ensuring that the construction is computable in $X \triangle A$. We build X by a finite extension argument, i.e., we construct an increasing sequence of strings $\sigma_0 \prec \sigma_1 \prec \cdots$ such that $X = \bigcup_{s\in\omega} \sigma_s$. During the construction, we will need to force the jump, i.e., we will make sure that at stage $s+1$, either $\{e\}^{\sigma_{s+1}}(e)\downarrow$, or there is no extension ρ of σ_{s+1} such that $\{e\}^{\rho}(e)\downarrow$. We will need to be able to compute from $X \triangle A$ which of the two cases applies in order to make the construction computable in $X \triangle A$.

To do this, we use a similar argument to the proof of a variant of Posner and Robinson's cupping theorem in Jockusch and Shore [3], where they use that A is either not c.e. or not co-c.e., except that in our case we use that A is either not almost c.e. or its complement is not almost c.e. Here, recall that a set A is *almost c.e.* if there is a computable approximation $(A_t)_{t\in\omega}$ to A such that for all n, if $A_t(n) = 1$ and $A_{t+1}(n) = 0$, then there is an $m < n$ with $A_t(m) = 0$ and $A_{t+1}(m) = 1$.[1] However, in case neither A nor its complement are almost c.e., to make the construction outlined in the next paragraph work, it turns out that we cannot fix one of these choices a priori, but we will need to make a choice "on the fly" while constructing each σ_{s+1}. In that case, we will use one extra bit to code this choice into $X \triangle A$.

However, even knowing which of the two cases applies is not enough to make the construction computable if we only have $X \triangle A$, because it turns out that the construction also depends on X, which is not automatically recoverable from $X \triangle A$. Fortunately, we already know that A is infinitely often K-trivial, so infinitely often we can code many of the bits of X into just a few bits of $X \triangle A$, i.e., we can code X into $X \triangle A$ in a very compact manner. Furthermore, as long as, after such a point n at which $A \restriction n$ is K-trivial, we have that $A \restriction m$ is uniformly computable from

[1]Note that almost c.e. sets are often called *left-c.e.*

$A \upharpoonright n$, the complexity cannot go up very much and therefore we can keep coding $A \upharpoonright n$ compactly. This combined with the ideas in the previous paragraph will yield the desired result.

Construction. Let $\sigma_0 = \emptyset$. To define σ_{s+1}, let n_{s+1} be least such that there exists a constant c for which both

$$|\sigma_s| + 2\log(n_{s+1}) + 2\log(m) + 3 + d + 2c \leq m$$

for all $m \geq n_{s+1}$,

$$K(A \upharpoonright n_{s+1}) \leq 2\log(n_{s+1}) + c,$$

and

$$K(k) \leq 2\log(k) + c$$

for all $k \geq 1$, where d is a constant which will be specified in the verification below. Note that we can find such an n_{s+1} because A is infinitely often K-trivial.

For $m \geq n_{s+1}$, let ρ_m be the first string of length $m - |\sigma_s| - 3$ such that $\mathcal{U}(\rho') = A \upharpoonright m$ for some $\rho' \preceq \rho_m$, if such a string ρ_m exists. For $a \in \{0,1\}$, let $\tau_m^a \in 2^{m+1}$ be the string

$$\sigma_s{}^\frown(A(|\sigma_s|) \triangle \emptyset'(s))^\frown(A(|\sigma_s|+1) \triangle a)^\frown((A \upharpoonright [|\sigma_s|+2, m-1)) \triangle \rho_m)^\frown 0.$$

Now, let $m_{s+1} \geq n$ be least such that $\rho_{m_{s+1}}$ and $\tau^a_{m_{s+1}}$ are defined and such that at least one of the following four cases holds:

$$m_{s+1} \notin A \wedge \exists \sigma \succeq \tau^0_{m_{s+1}}(\{s\}^\sigma(s)\downarrow) \tag{1}$$

$$m_{s+1} \in A \wedge \forall \sigma \succeq \tau^0_{m_{s+1}}(\{s\}^\sigma(s)\uparrow) \tag{2}$$

$$m_{s+1} \in A \wedge \exists \sigma \succeq \tau^1_{m_{s+1}}(\{s\}^\sigma(s)\downarrow) \tag{3}$$

$$m_{s+1} \notin A \wedge \forall \sigma \succeq \tau^1_{m_{s+1}}(\{s\}^\sigma(s)\uparrow). \tag{4}$$

(We will argue below that m_{s+1} always exists.) Let i be the first number for which we find that (i) holds for m_{s+1}. If (i) is (1), let σ_{s+1} be the least such $\sigma \succeq \tau^0_{m_{s+1}}$. If (i) is (2), let σ_{s+1} be $\tau^0_{m_{s+1}}$. If (i) is (3), let σ_{s+1} be the least such $\sigma \succeq \tau^1_{m_{s+1}}$, and finally, if (i) is (4), let σ_{s+1} be $\tau^1_{m_{s+1}}$.

Verification. Our first claim is that at every stage $s+1$, there is a number m_{s+1} as described. Towards a contradiction, assume otherwise. First, assume that ρ_m is defined for all $m \geq n_{s+1}$; hence, τ_m^a is defined for $a \in \{0,1\}$. We claim that both A and its complement are almost c.e. and hence A is computable, which would be a contradiction. To this end, let $A_t(m) = A(m)$ for $m < n_{s+1}$, let $A_t(m) = 1$ for $m \geq n$ if there exists a string $\sigma \succeq \tau_m^0[t]$ of length at most t such that $\{s\}^\sigma(s)\downarrow$, and let $A_t(m) = 0$ otherwise. Here, $\tau_m^0[t]$ is defined as τ_m^0 above, except we replace all occurrences of A in its definition by $A_t \upharpoonright m$.

Then $(A_t)_{t\in\omega}$ is a computable approximation to A witnessing that A is almost c.e. First, we can prove that it converges to A by induction on m. If $m < n_{s+1}$, then this is clear. If $m \geq n_{s+1}$, then since neither case (1) nor case (2) hold, we know that $m \in A$ if and only if $\exists \sigma \succeq \tau_m^0(\{s\}^\sigma(s)\downarrow)$. By induction, we know that $A_t \upharpoonright m = A \upharpoonright m$ for large enough t, and hence $\tau_m^0[t] = \tau_m^0$ for large enough t, since the definition of $\tau_m^0[t]$ only depends on $A_t \upharpoonright m$. Thus, if $m \in A$, then $m \in A_t$ for almost all t, and if $m \notin A$, then $m \notin A_t$ for almost all t, as desired.

Furthermore, if at some stage $t+1$ there is an $m \in A_t \setminus A_{t+1}$, let m be the least such element. Then this can only be because $\tau_m^1[t+1] \neq \tau_m^1[t]$, and hence $A_t \upharpoonright m \neq A_{t+1} \upharpoonright m$. Thus, there is a $m' < m$ such that $A_t(m') \neq A_{t+1}(m')$, and by minimality of m we therefore have that $m' \in A_{t+1} \setminus A_t$, as desired.

On the other hand, to show that $B = \omega \setminus A$ is almost c.e., let $B_t(m) = B(m)$ for $m < n_{s+1}$, let $B_t(m) = 1$ for $m \geq n$ if there exists a string $\sigma \succeq \tau_m^1[t]$ of length at most t such that $\{s\}^\sigma(s)\downarrow$, and let $B_t(m) = 0$ otherwise. Then $(B_t)_{t \in \omega}$ witnesses that B is almost c.e. by a similar argument as above, using the fact that (3) and (4) do not hold for any $m \geq n$.

Therefore, some ρ_k is not defined; we let $k \geq n_{s+1}$ be least such that ρ_k is not defined. Thus, there is no string ρ with $\mathcal{U}(\rho') = A \upharpoonright k$ of length at most $k - |\sigma_s| - 3$. However, then the argument above still works up to $k-1$; that is, the argument above gives us a single algorithm which, on input $A \upharpoonright n_{s+1}$ and m, with $n_{s+1} \leq m < k$, outputs $A(m)$. Therefore, there is a constant d, independent of A, n_{s+1} and k, such that

$$K(A \upharpoonright k) \leq K(A \upharpoonright n_{s+1}) + K(k) + d.$$

Then, we have that

$$K(A \upharpoonright k) \leq 2\log(n_{s+1}) + 2c + 2\log(k) + d \leq k - |\sigma_s| - 3,$$

which is a contradiction. Therefore, there is a number m_{s+1} as required by the construction.

Note that X is clearly 1-generic because it is enough to force the jump. We also claim that the construction is computable in $X \triangle A$, and therefore $\emptyset'$ is clearly computable from $X \triangle A$. Indeed, given σ_s, to determine σ_{s+1}, let m' be the unique number such that $\mathcal{U}((X \triangle A) \upharpoonright [|\sigma_s| + 2, m' - 1))\downarrow$; then by construction, the output of this has to be $A \upharpoonright m_{s+1}$. Also, let $a = (X \triangle A)(|\sigma_s| + 1)$, and finally, let $b = (X \triangle A)(m_{s+1})$. Then a and b allow us to determine which of the four cases we took in the construction of σ_{s+1}: if $a = 0$ we took case (1) or (2), and if $a = 1$ we took case (3) or (4). Furthermore, we took case (1) or (4) if $b = 0$, and we took case (2) or (3) if $b = 1$. It is now not hard to see how to compute σ_{s+1}. □

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