ON WEIHRAUCH REDUCIBILITY AND INTUITIONISTIC REVERSE MATHEMATICS

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ABSTRACT. We show that there is a strong connection between Weihrauch reducibility on one hand, and provability in EL_0, the intuitionistic version of RCA_0, on the other hand. More precisely, we show that Weihrauch reducibility to the composition of finitely many instances of a theorem is captured by provability in EL_0 together with Markov's principle, and that Weihrauch reducibility is captured by an affine subsystem of EL_0 plus Markov's principle.

1. INTRODUCTION

There are two main approaches to classifying the relative strength of theorems in mathematics, where we are usually interested in statements which can be formulated as Π^1_2-formulas, i.e. formulas of the form $\forall X \xi(X) \to \exists Y \psi(X, Y)$. One of them, in the form of *reverse mathematics*, classifies theorems by looking at their relative strength over a weak system, RCA_0. For more background on reverse mathematics we refer to Simpson [13].

The other approach, in the form of *Weihrauch reducibility*, classifies theorems by their uniform computational power. In this case, we say that ζ_0 is Weihrauch reducible to ζ_1 if we can uniformly, computably transform instances of ζ_0 into instances of ζ_1, and given any solution for this instance of ζ_1 we can uniformly compute a solution to ζ_0 for the original instance from ζ_1 and this original instance. For more background on Weihrauch reducibility we refer to Brattka and Gherardi [1].

Until recently, research in these two fields has been mostly disjoint. Dorais, Dzhafarov, Hirst, Mileti and Shafer [5] and Dzhafarov [7] try to bring together these two fields, by formalising Weihrauch reducibility within RCA_0. This shows that Weihrauch-reducibility within RCA_0 is, in a certain sense, a special case of reverse mathematics, namely the case where one has to prove ζ_0 uniformly using only one application of ζ_1.

On the other hand, several people have studied the connections between provability of sequential versions of Π^1_2-formulas in RCA_0, and provability in EL_0. This research was started by Hirst and Mummert [9], and continued by Dorais [6] and Fujiwara [8]. Their proofs use techniques from realisability, a field which has long studied the connections between intuitionistic logic and computability.

In the current paper, we combine these two research directions to show that there is a tight connection between Weihrauch reducibility and provability in EL_0. Recall that, in the Weihrauch degrees, there is a natural notion of composition, denoted by $\star$, which was introduced by Brattka, Gherardi and Marcone [2] and shown to exist by Brattka, Oliva and Pauly [3]. Here, ζ_0 is Weihrauch-reducible to $\zeta_2 \star \zeta_1$ if and only if we can solve ζ_0 by transforming an instance X of ζ_0 into an instance Y

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of ζ_1, and then given a solution for Y, transforming this into an instance Y' for ζ_2, and then given any solution for Y', transforming this into a solution for X, all in a computable and uniform way. Note that there are three computable transformations taking place, and hence such a reduction is witnessed by three indices e_1, e_2 and e_3.

The main theorem of this paper is that there is some computable transformation $\zeta \mapsto \zeta'$ of Π^1_2-formulas satisfying that ζ is classically equivalent to ζ', for which the following are equivalent:

(1) There are an $n \in \omega$ and $e_1, \dots, e_{n+1}$ such that RCA_0 proves that $e_1, \dots, e_{n+1}$ are indices for Turing functionals witnessing that ζ_0 Weihrauch-reduces to the composition of n copies of ζ_1.
(2) $\mathrm{EL}_0 + \mathrm{MP}$ proves that $\zeta_1' \to \zeta_0'$.

Here, recall that MP is *Markov's principle*, that is, $\neg\neg\exists x\phi \to \exists x\phi$ for atomic formulas ϕ.

While this result shows that intuitionistic provability is very close to Weihrauch reducibility, in the sense that it captures the uniformity expressed by Weihrauch reducibility, it does not capture the other aspect which makes Weihrauch reducibility special, namely that it only allows one to use one instance of ζ_1. However, in this paper we also prove that, if we suitably restrict our sequent calculus to an affine version $\mathrm{IQC}^{\exists\alpha\mathrm{a}}$, to be defined below, we get the following result:

(1) There are e_1, e_2 such that RCA_0 proves that e_1, e_2 witness that ζ_0 Weihrauch-reduces to ζ_1.
(2) $(\mathrm{EL}_0 + \mathrm{MP})^{\exists\alpha\mathrm{a}}$ proves that $\zeta_1' \to \zeta_0'$.

This shows that Weihrauch-reducibility can be seen as a proof-theoretic notion that is stronger than RCA_0. That is, proving that ζ_0 is Weihrauch-reducible to ζ_1 in RCA_0 can be seen as proving that ζ_0' implies ζ_1' using a restricted set of derivation rules.

The paper is structured as follows. In the next section, we will introduce the sequent calculus for intuitionistic predicate logic, or IQC, that we will be using throughout this paper, and we will introduce EL_0. Next, in section 3 we will briefly discuss Weihrauch reducibility and how we formalise this notion within EL_0. After that, in section 4 we will introduce a specially tailored reducibility notion, which will allow us to extract Weihrauch reductions from proofs in EL_0. Then, in section 5 we show how this extraction works, and therefore prove the first implication of the first main result mentioned above. The converse of this is proven in section 6. Finally, in the last section we show how to capture Weihrauch reducibility using $\mathrm{IQC}^{\exists\alpha\mathrm{a}}$.

Our notation is mostly standard. We let ω denote the (standard) natural numbers. For undefined notions from proof theory, we refer to Buss [4].

2. IQC and EL_0

In this section, we will formalise the sequent calculus for IQC that we will be working with in this paper, since the proofs depend on the exact calculus we take. After that, we will discuss EL_0 and the affine version mentioned in the introduction.

Definition 2.1. We say that a sequent $\Gamma \vdash \phi$ is derivable in IQC if it is derivable using the following rules.

$$\frac{}{A \vdash A}\,(I)$$

$$\frac{\Gamma, \psi_1, \psi_2 \vdash \phi}{\Gamma, \psi_1 \wedge \psi_2 \vdash \phi}\ (\wedge L) \qquad \frac{\Gamma_1 \vdash \phi_1 \qquad \Gamma_2 \vdash \phi_2}{\Gamma_1, \Gamma_2 \vdash \phi_1 \wedge \phi_2}\ (\wedge R)$$

$$\frac{\Gamma_1 \vdash \psi_1 \qquad \Gamma_2, \psi_2 \vdash \phi}{\Gamma_1, \Gamma_2, \psi_1 \to \psi_2 \vdash \phi}\ (\to L) \qquad \frac{\Gamma, \phi_1 \vdash \phi_2}{\Gamma \vdash \phi_1 \to \phi_2}\ (\to R)$$

$$\frac{\Gamma, \psi[x := t] \vdash \phi}{\Gamma, \forall x \psi \vdash \phi}\ (\forall L) \qquad \frac{\Gamma \vdash \phi[x := y]}{\Gamma \vdash \forall x \phi}\ (\forall R)$$

$$\frac{\Gamma, \psi[x := y] \vdash \phi}{\Gamma, \exists x \psi \vdash \phi}\ (\exists L) \qquad \frac{\Gamma \vdash \phi[x := t]}{\Gamma \vdash \exists x \phi}\ (\exists R)$$

$$\frac{\Gamma \vdash \phi}{\Gamma, \psi \vdash \phi}\ (W) \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash A}\ (\bot)$$

$$\frac{\Gamma, \psi, \psi \vdash \phi}{\Gamma, \psi \vdash \phi}\ (C)$$

$$\frac{\Gamma_1, \psi_1, \psi_2, \Gamma_2 \vdash \phi}{\Gamma_1, \psi_2, \psi_1, \Gamma_2 \vdash \phi}\ (P)$$

$$\frac{\Gamma_1 \vdash \psi \qquad \Gamma_2, \psi \vdash \phi}{\Gamma_1, \Gamma_2 \vdash \phi}\ (\mathrm{Cut})$$

Here, A is an atomic formula, y does not appear free in Γ and ϕ in $(\exists L)$, and y does not appear free in Γ in $\forall R$. Furthermore, in case ψ_2 in $(\to L)$ is $\bot$, we only allow this rule if ϕ is also $\bot$ (we will also call this rule $(\neg L)$).

As commonly done in intuitionistic logic, we do not have any rules for $\neg$, because $\neg$ is not a primitive connective: $\neg\phi$ is interpreted as $\phi \to \bot$. However, note that our calculus also does not contain any rules for $\vee$. This is because, as is usual in EL and EL_0, we do not see $\vee$ as a primitive connective either, but instead define $\phi \vee \psi$ as $\exists x((x = 0 \to \phi) \wedge (x \neq 0 \to \psi))$. This is justified by the fact that equality is decidable in EL_0, see Troelstra [14]. Furthermore, in EL this definition for $\vee$ is equivalent to the usual intuitionistic disjunction, see [14, section 1.3.7].

However, it is important to note that this definition of $\vee$ is *weaker* than the usual intuitionistic disjunction, because the proof of their equivalence uses arithmetical induction, which we do not have available in EL_0. We will come back to this in Remark 2.4 below, after we have formalised EL_0.

Let us adopt the following standard convention: *every variable is bound at most once, and no variable can be both bound and free simultaneously.* Furthermore, let us adopt the convention that once a bound variable disappears from a proof because of an application of the cut rule, then we are not allowed to reuse that variable later in the proof.

Now, let us define EL_0, following Dorais [6] (who mostly follows [14]). Unlike the usual formulation of RCA_0, this is done using functions instead of sets. Let us denote number variables by $x, y, \dots$ and function variables by $\alpha, \beta, \dots$. The language contains the constant 0, the successor function S, equality, and a function symbol for every primitive recursive function. There are two kinds of terms: number terms t and function terms τ. They are formed as follows:

- Every number variable is a number term.
- Every function variable is a function term.
- The zero constant 0 is a number term.
- If t is a number term, then so is $S(t)$.
- If $t_1, \dots, t_n$ are number terms and f is a symbol for an n-ary primitive recursive function, then $f(t_1, \dots, t_n)$ is a number term.

- If t is a number term and τ is a function term, then the evaluation $\tau(t)$ is a number term.
- If t is a number term and x is a number variable, then the lambda abstraction $\lambda x.t$ is a function term.
- If t is a number term and γ is a function term, then the recursion $\mathrm{R}t\tau$ is a function term.

We only have one atomic relation in our language: equality on the number sort. Equality on the function sort is defined by extensionality, i.e.

$$\tau_1 = \tau_2 \leftrightarrow \forall x(\tau_1(x) = \tau_2(x)).$$

Definition 2.2. Let $\Gamma \vdash \phi$ be a sequent in the language described above. Then we say that $\Gamma \vdash \phi$ is derivable in EL_0 if it is derivable in IQC using the equality axioms, the defining axioms for all the primitive recursive function symbols, and the following axioms.

(SA) $$S(x) \neq 0 \wedge (S(x) = S(y) \to x = y)$$

(QF-IA) $$\forall x(B(x) \to B(S(x))) \to \forall y(B(0) \to B(y))$$

(CON) $$(\lambda x.t)(t') = t[x := t']$$

(REC) $$(\mathrm{R}t\tau)(0) = t \wedge (\mathrm{R}t\tau)(S(t')) = \tau((\mathrm{R}t\tau)(t'))$$

($\mathrm{QF\text{-}AC}^{0,0}$) $$\forall x \exists y B(x,y) \to \exists \alpha \forall z B(z, \alpha(z))$$

Here, B is a quantifier-free formula.

Since our system has symbols for all primitive recursive functions, we can encode finite sequences of numbers in the usual manner. We will implicitly do this throughout this paper.

Although RCA_0 is usually formulated using sets instead of functions, there is a natural connection between them, since we can identify a function with its graph, and a set with its indicator function. Indeed, if we add the *law of the excluded middle*, i.e. $\phi \vee \neg\phi$, to EL_0, we get a system that is equivalent to RCA_0 under the identification just described (see Dorais [6] or Kohlenbach [11]). A particular ingredient of this proof is that EL_0 in fact allows induction (QF-IA) and choice ($\mathrm{QF\text{-}AC}^{0,0}$) for not just quantifier-free formulas B, but even for Σ^0_1-formulas ϕ.

Proposition 2.3. *Let $\phi(x) = \exists y B(x,y)$, where B is quantifier-free. Then* EL_0 *proves*

$$\forall x(\phi(x) \to \phi(S(x))) \to \forall x(\phi(0) \to \phi(x))$$

and

$$\forall x \exists y \phi(x,y) \to \exists \alpha \forall x \phi(x, \alpha(x)).$$

Proof. Consider the relation $C(x,y)$ which holds if and only if both y is a sequence of length $x+1$, and for every $z \leq x$ we have that $B(x, y(z))$ holds. Note that this relation is primitive recursive in B, so using the recursion operator we can find a function term τ representing the indicator function of this relation.[1] Let us reason within EL_0. Assume $\forall x(\phi(x) \to \phi(S(x)))$ and $\phi(0)$ both hold. Then, using QF-IA,

[1]Note that B might contain function variables, so we cannot directly use the fact that the language contains a symbol for every primitive recursive function.

we know that $\forall x \exists y(\tau(x,y)=1)$ holds. Thus, by QF-AC^{0,0} there is a function f such that $\forall x(\tau(x,f(x))=1)$ holds. In particular, we have that $\forall x(B(x,f(x)(x)))$ holds, and therefore $\forall x(\phi(x))$ holds.

That EL_0 proves the axiom of choice for Σ^0_1-formulas follows directly using a pairing function. □

Remark 2.4. As noted above, we do not see $\vee$ as a primitive connective, but instead interpret $\phi \vee \psi$ as $\exists x((x=0 \to \phi) \wedge (x \neq 0 \to \psi))$. We also noted that this interpretation is equivalent to the usual intuitionistic disjunction in EL, i.e. EL_0 with full induction. However, in EL_0 this is not true, because we do not have enough induction to prove this. In EL_0 only a weak version of the usual $\vee L$ rule holds: we generally only have that

$$\frac{\Gamma, \psi_1 \vdash \phi \qquad \Gamma, \psi_2 \vdash \phi}{\Gamma, \psi_1 \vee \psi_2 \vdash \phi}$$

holds for Σ^0_1-formulas ϕ, and for formulas ϕ starting with a negation. In fact, our proofs implicitly exploit this fact. Fortunately, all the other usual logical laws, such as the commutativity of $\vee$, still hold.

There is one additional principle which, although not part of EL_0, will be used in this paper.

Definition 2.5. *Markov's principle*, or MP, is the principle

$$\neg\neg\exists x B(x) \to \exists y B(y)$$

for quantifier-free formulas B.

As for RCA_0, much of elementary computability theory can be carried out inside EL_0; see Kleene [10][2] and Troelstra [14] for many details.

In particular, the normal form theorem holds, in the following form. Here, $\alpha_1 \oplus \cdots \oplus \alpha_n$ denotes the function which sends $kn+i$ to $\alpha_{i-1}(k)$, and just like for the pairing functions for numbers, EL_0 proves all relevant properties about this join. Below, $\ulcorner e \urcorner$ denotes the term representing the number e.

Theorem 2.6. ([10]) *There exists a quantifier-free formula $\phi(y_1,y_2,\beta,y_3,y_4,y_5)$ and a term $s(z)$ such that for every function term $\tau(\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_m)$ there exists an $e \in \omega$ such that in EL_0 we have:*

$$\begin{aligned}
&\forall \alpha_1,\ldots,\alpha_n \forall x_1,\ldots,x_m \forall a,b\\
&\quad \tau(\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_m)(a)=b\\
&\qquad \leftrightarrow \exists k(\phi(\ulcorner e \urcorner, \langle n,m\rangle, \alpha_1 \oplus \cdots \oplus \alpha_n, \langle x_1,\ldots,x_m\rangle, a, k) \wedge s(k)=b)\\
&\qquad \leftrightarrow \forall k(\phi(\ulcorner e \urcorner, \langle n,m\rangle, \alpha_1 \oplus \cdots \oplus \alpha_n, \langle x_1,\ldots,x_m\rangle, a, k) \to s(k)=b).
\end{aligned}$$

Proof. This follows from Lemma 41 and the discussion on p. 69 of [10]. □

In view of this proposition, we will write

$$\Phi_e(\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_m)(a)=b$$

for the formula

$$\exists k(\phi(\ulcorner e \urcorner, \langle n,m\rangle, \alpha_1 \oplus \cdots \oplus \alpha_n, \langle x_1,\ldots,x_m\rangle, a, k) \wedge s(k)=b).$$

Also, we will write

$$\Phi_e(\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_m)=\beta$$

for the formula

$$\forall a(\Phi_e(\alpha_1,\ldots,\alpha_n,x_1,\ldots,x_m)(a)=\beta(a)).$$

[2] Although Kleene uses a different system, as mentioned in Troelstra [14, p. 73], his developments apply to our system as well.

Finally, we will write

$$\Phi_e(\alpha_1, \dots, \alpha_n, x_1, \dots, x_m) = c$$

for the formula

$$\Phi_e(\alpha_1, \dots, \alpha_n, x_1, \dots, x_m)(0) = c.$$

For some of our results below, we need to look at a weaker, affine version of IQC. Here, by affine we mean that we restrict the use of the contraction rule (C). Alternatively, we could have formalised this system by a suitable embedding into linear logic, but we think the current approach is more intuitive.

Definition 2.7. We say that a sequent $\Gamma \vdash \phi$ is derivable in $\mathrm{IQC}^{\mathrm{a}}$ if it is derivable using the rules from Definition 2.1, except for (C).

Finally, let us define two affine versions of EL_0.

Definition 2.8. We define $\mathrm{EL}_0^{\mathrm{a}}$ as in Definition 2.2, except using $\mathrm{IQC}^{\mathrm{a}}$ instead of IQC. Furthermore, we say that a sequent $\Gamma \vdash \phi$ is derivable in $\mathrm{EL}_0^{\exists\alpha\mathrm{a}}$ if it is derivable from the axioms in Definition 2.2 using the rules from Definition 2.1, except we only allow (C) for formulas not containing function quantifiers, and we only allow (W) for subformulas of formulas of the form $\exists\alpha\psi$ where ψ does not contain any function quantifiers. We define $(\mathrm{EL}_0 + \mathrm{MP})^{\mathrm{a}}$ and $(\mathrm{EL}_0 + \mathrm{MP})^{\alpha\mathrm{a}}$ in a similar way.

3. Weihrauch reducibility

In this section we will discuss how Weihrauch reducibility can be formalised within EL_0. Weihrauch reducibility is normally defined using represented spaces, see e.g. Brattka and Gherardi [1]. However, we are specifically looking at problems given by formulas of the form $\forall\alpha\xi(\alpha) \to \exists\beta\psi(\alpha,\beta)$ (which we will henceforth call Π^1_2-formulas), in which case Weihrauch reducibility can be defined in an easier way (see also Dorais et al. [5] and Dzhafarov [7]). So, let us define Weihrauch reducibility within EL_0 as follows.

Definition 3.1. Let $\zeta_0 = \forall\alpha_0\xi_0(\alpha_0) \to \exists\beta_0\psi_0(\alpha_0,\beta_0)$ and $\zeta_1 = \forall\alpha_1\xi_1(\alpha_0) \to \exists\beta_1\psi_1(\alpha_1,\beta_1)$ be Π^1_2-formulas. Then we define, in EL_0, that ζ_0 *Weihrauch-reduces to* ζ_1 if there exists e_1, e_2 such that

$$\begin{aligned}&\forall\alpha_0(\xi_0(\alpha_0) \to \Phi_{e_1}(\alpha_0)\downarrow \wedge\, \xi_1(\Phi_{e_1}(\alpha_0)))\\ \wedge&\forall\alpha_0\forall\beta_0((\xi_0(\alpha_0) \wedge \psi_1(\Phi_{e_1}(\alpha_0),\beta_0))\\ &\quad\to \Phi_{e_2}(\alpha_0,\Phi_{e_1}(\alpha_0),\beta_0)\downarrow \wedge\, \psi_0(\alpha_0,\Phi_{e_2}(\alpha_0,\Phi_{e_1}(\alpha_0),\beta_0))).\end{aligned}$$

As discussed in the introduction, there is a notion of composition, the *compositional product*, on Weihrauch degrees. We would like to define the composition of two Π^1_2-formulas in a way which corresponds to this notion in the Weihrauch degrees, but it does not seem like this can be done using a Π^1_2-formula. Instead, we define what it means to be reducible to a composition.

Definition 3.2. Let $\zeta_0 = \forall\alpha_0\xi_0(\alpha_0) \to \exists\beta_0\psi_0(\alpha_0,\beta_0), \dots, \zeta_n = \forall\alpha_n\xi_n(\alpha_n) \to \exists\beta_n\psi_n(\alpha_n,\beta_n)$ be Π^1_2-formulas. Then we define, in EL_0, that ζ_0 *Weihrauch-reduces to the composition of* $\zeta_n, \dots, \zeta_1$ if there exist an $n \geq 0$ and $e_1, \dots, e_{n+1}$ such that, for all α_0 and for all $\beta_1, \dots, \beta_n$, and for every $1 \leq i \leq n$, we have that, if for $1 \leq j \leq n$ we write α_j for $\Phi_{e_j}(\alpha_0, \alpha_1, \dots, \alpha_{j-1}, \beta_1, \dots, \beta_{j-1})$, then, if for all $1 < j \leq i$ we have that $\psi_j(\alpha_j,\beta_j)$ holds, and $\xi_0(\alpha_0)$ holds, then

$$\Phi_{e_i}(\alpha_0, \alpha_1, \dots, \alpha_{i-1}, \beta_1, \dots, \beta_{i-1})\downarrow$$

holds and $\xi_i(\alpha_i)$ holds; and finally, that if for all $1 \leq j \leq n$ we have that $\psi_j(\alpha_j,\beta_j)$ holds, and $\xi_0(\alpha_0)$ holds, then we have that

$$\Phi_{e_{n+1}}(\alpha_0, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)\downarrow$$

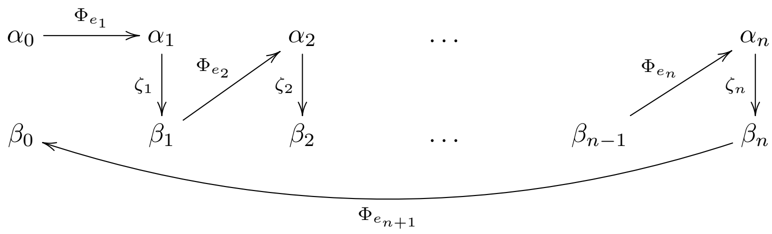

FIGURE 1. A Weihrauch reduction of ζ_0 to the composition of $\zeta_n, \dots, \zeta_1$.

holds, and that

$$\psi_0(\alpha_0, \Phi_{e_{n+1}}(\alpha_0, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n))$$

holds.

We say that $e_1, \dots, e_{n+1}$ *witness* the Weihrauch reduction.

It is not hard to verify that ζ_0 Weihrauch-reduces to the composition of ζ_1 if and only if ζ_0 Weihrauch-reduces to ζ_1, as expected.

If we see each $\zeta_n, \dots, \zeta_1$ as a black box transforming instances into solutions, Figure 1 illustrates what it means for ζ_0 to Weihrauch-reduce to the composition of $\zeta_n, \dots, \zeta_1$.[3]

4. Realisability

To prove our result, we will introduce a specially tailored notion of realisability. Our realisers will be pairs of functions of the form $v, w : \{1, \dots, n\} \to \omega \cup (\omega \times \omega)$. Let us explain the motivation behind them. First, let us consider w. The idea is that, if $w(i) \in \omega \times \omega$, say $w(i) = (k, e)$, then w says that the intended value for x_i can be computed using the Turing functional Φ_e. Similarly, if $v(i) = (k, e)$, then v says that the intended value for α_i can be computed using the Turing functional Φ_e. Furthermore, we want to carefully bookkeep which other variables it is allowed to use in the computation, which is what we use k for. Let us define $w_1(j)$ as the first coordinate of $w(j)$ if $w(j) \in \omega \times \omega$, and as $w(j)$ otherwise. We define v_1 in the same way.

Then, if $w(i) = (k, e)$, we can compute x_i using Φ_e, but we are only allowed to use those α_i with $v_1(\alpha_i) < k$ and those x_j with $w_1(j) < k$. Similarly, if $v(i) = (k, e)$, then we can compute the intended value for α_i using α_j with $v_1(j) < k$, and x_j with $w_1(j) < k$.

Let us formalise this idea.

Definition 4.1. Let $v : \{1, \dots, n\} \to \omega \cup (\omega \times \omega)$ and $w : \{1, \dots, n'\} \to \omega \cup (\omega \times \omega)$. We say that a sequence of functions $f_1, \dots, f_n$ and a sequence of numbers $a_1, \dots, a_{n'}$ are *valid for* (v, w) *at* $v(m)$ if, whenever $v(m) \in \omega \times \omega$, say $v(m) = (k, e)$, if $i_1, \dots, i_s$ lists in increasing order those $1 \le i \le n$ with $v_1(i) < k$, and $j_1, \dots, j_t$ lists in increasing order those $1 \le j \le n'$ with $w_1(j) < k$, then we have that, if $\Phi_e(f_{i_1}, \dots, f_{i_s}, a_{j_1}, \dots, a_{j_t})\downarrow$, then it is equal to f_m. We say that it is *strongly valid for* (v, w) *at* $v(m)$ if it is both valid for (v, w) at $v(m)$, and $\Phi_e(f_{i_1}, \dots, f_{i_s}, a_{j_1}, \dots, a_{j_t})\downarrow$.

Similarly, we define the notions of valid and strongly valid for (v, w) at $w(m)$ using a_m.

Now, we say that $(f_1, \dots, f_n, a_1, \dots, a_n)$ is *valid for* (v, w) if it is valid at $v(m)$ for all $1 \le m \le n$ and it is valid for $w(m)$ at all $1 \le m \le n'$.

[3]Note that Φ_{e_i} is allowed to use all of the functions appearing earlier in the sequence. For example, Φ_{e_2} is allowed to use α_0, α_1 and β_1, and not just β_1, although the diagram might suggest otherwise.

Finally, if it is true that whenever $v(m) = (k, e)$ or $w(m) = (k, e)$ for some m, we have for all i with $v(i) \in \omega \times \omega$ that the computation of the functional Φ_e is independent of f_i (i.e. it ignores the input f_i), and we also have for all j with $w(j) \in \omega \times \omega$ that the computation is independent of a_j, then we will say that (v, w) is *self-contained.*

Note that we can identify

$$\bigcup_{n,n' \in \omega} (\{1, \ldots, n\} \to \omega \cup (\omega \times \omega)) \times (\{1, \ldots, n'\} \to \omega \cup (\omega \times \omega))$$

with the set of natural numbers in a primitive recursive way, and hence in a way which is definable within EL_0. We will tacitly assume this. Given any v, we let $|v|$ be the number n such that v is defined on $\{1, \ldots, n\}$, and similarly for w.

In our proofs, we will be considering Π^1_2-formulas. Let us therefore make the following definition.

Definition 4.2. A Π^1_2-sequent is a sequent $\Gamma \vdash \phi$ such that all formulas in Γ and ϕ are Π^1_2, i.e. of the form $\forall \alpha \xi(\alpha) \to \exists \beta \psi(\alpha, \beta)$ where ψ and ξ are arithmetical.

To make the proof of our main theorem work, we will need to make the contractions used in the proof explicit. For this, we use the following definition, together with two lemmas.

Definition 4.3. Let ϕ, ψ be formulas. We say that ϕ is *contraction-similar to* ψ if there exists an $n \geq 0$ and a sequence $\psi = \phi_0, \phi_1, \ldots, \phi_n = \phi$ such that for every $1 \leq i \leq n$, ϕ_i is obtained from ϕ_{i-1} by replacing a subformula χ of ϕ_{i-1} by $\chi_1 \wedge \chi_2$, where χ_1 and χ_2 are both contraction-similar to χ.

Furthermore, we say that a finite set Σ of formulas is *contraction-similar* to a finite set Γ of formulas if there exists a surjection f of Σ onto Γ such that every formula ϕ in Σ is contraction-similar to $f(\phi)$. We will say that ϕ and $f(\phi)$ are *linked.*

Finally, a sequent $\Sigma \vdash \phi$ is *contraction-similar to* $\Gamma \vdash \psi$ if Σ is contraction-similar to Γ and ϕ is contraction-similar to ψ.

The proof of the first lemma is a straightforward proof by induction on ϕ.

Lemma 4.4. *If ϕ is contraction-similar to ψ, then $\phi \leftrightarrow \psi$ is provable in* IQC.

On the other hand, as long as we replace our sequents by suitable contraction-similar sequents, we can transfer proofs from EL_0 to $\mathrm{EL}_0^{\mathrm{a}}$. Below we will, however, also need that the proofs can be sufficiently cut free. Recall that a proof is *free-cut free* if the principal formulas of any cut appearing in the proof are axioms, e.g. a proof in EL_0 is free-cut free if every principal formula of a cut appearing in the proof is one of the axioms from Definition 2.2.

Lemma 4.5. *Let Σ be the set of formulas which are contraction-similar to an axiom of* EL_0*. If a Π^1_2-sequent $\Gamma \vdash \phi$ is provable in* EL_0*, then there is a free-cut free proof in* $\mathrm{EL}_0^{\mathrm{a}}$ *from Σ of some Π^1_2-sequent $\Gamma' \vdash \phi'$ which is contraction-similar to $\Gamma \vdash \phi$. The same holds if we replace* EL_0 *by* $\mathrm{EL}_0 + \mathrm{MP}$ *and* $\mathrm{EL}_0^{\mathrm{a}}$ *by* $(\mathrm{EL}_0 + \mathrm{MP})^{\mathrm{a}}$.

Proof. We prove this by induction on a free-cut free proof P of $\Gamma \vdash \phi$ in EL_0. The core idea is that, whenever we have an application of the contraction rule in P, we replace this by an instance of $(\wedge L)$ instead. This would suffice if we did not require $\Gamma' \vdash \phi'$ to be a Π^1_2-sequent, but to ensure that it is a Π^1_2-sequent we cannot apply this replacement if the principal formula ψ contains a function quantifier (or, equivalently, an existential function quantifier). Instead, in this case we completely remove the instance of the contraction rule, without replacing it with anything else. This means that in the conclusion, we now have two copies ψ_1, ψ_2 of ψ, and hence we need to modify the lower steps in the original proof P. This is simple to do: we *link* ψ_1 and ψ_2 to ψ, and henceforth, whenever some rule was applied to ψ in P, we apply two copies of it to both ψ_1 and ψ_2.

Let us give the details. It turns out that we need to slightly strengthen the claim to make the induction work: we add the requirement that, if any formula does not contain an existential function quantifier, then it is linked to at most one formula. First, let us consider the most crucial case, which is when the last step in the proof is an instance of the contraction rule. Thus, we have a proof of $\Gamma, \psi, \psi \vdash \phi$. Let us first assume that ψ does not contain an existential function quantifier. By the induction hypothesis, we therefore have a proof of some contraction-similar sequent $\Gamma', \zeta, \xi \vdash \phi'$ where ζ and ξ are contraction-similar to ψ. But then we also have a proof of $\Gamma', \zeta \wedge \xi \vdash \phi'$ (renaming some bound variables, if necessary), and this sequent is contraction-similar to $\Gamma, \psi \vdash \phi$. On the other hand ψ does contain an existential function quantifier, then $\Gamma', \zeta, \xi \vdash \phi'$ is already contraction-similar to $\Gamma, \psi \vdash \phi$, if we declare ζ and ξ to be linked to ψ.

Next, if we are in the case where the last step of the proof P is an instance of some right-side rule, we can just apply the same rule in our new proof P'. For example, if it is an instance of $(\to R)$, with conclusion $\Gamma \vdash \phi_1 \to \phi_2$, then the induction hypothesis gives us a proof of $\Gamma', \phi_1' \vdash \phi_2'$ in $\mathrm{EL}_0^{\mathrm{a}}$ of a contraction-similar sequent. Here, we let ϕ_1' be the unique formula which is contraction-similar to ϕ_1, which follows from our strengthened induction hypothesis and the fact that ϕ_1' cannot contain an existential function quantifier (since P is free-cut free). Now an application of $(\to R)$ gives us $\Gamma' \vdash \phi_1' \to \phi_2'$, which is contraction-similar to $\Gamma \vdash \phi_1 \to \phi_2$.

Now, let us consider what happens if the last step in the proof is an instance of some left-side rule. For example, let us consider what happens if this is an instance of $(\forall \alpha_i L)$. In this case, list the formulas $\zeta_1, \dots, \zeta_n$ which are linked to the principal formula ψ, and apply an instance of the $\forall \alpha_i L$ rule to each ζ_j. Now, link each $\forall \alpha_i \zeta_j$ to $\forall \alpha_i \psi$.

Finally, there is one rule which falls under neither of these cases, which is the cut rule. Since our proof is free-cut free, this can only happen if the cut formula ψ is an axiom of EL_0. All the formulas $\zeta_1, \dots, \zeta_n$ linked to ψ are therefore elements of Σ, which means we can replace this single instance of the cut rule by n axioms of the form $\vdash \zeta_j$, and n many instances of the cut rule, with $\zeta_1, \dots, \zeta_n$ as their cut formulas. ☐

We are now going to define our realisability notion. That is, we are going to define a binary formula

$$(v, w)\,\mathrm{real}\,\Gamma \vdash \phi.$$

To do so, we first need to define what it means to be realised at some sequence $(f_1, \dots, f_{|v|}, a_1, \dots, a_{|w|})$, of which we will need a positive and negative version. We do this by recursion on the formulas (which we can do in EL_0, using the recursion operator).

Definition 4.6. We simultaneously define

$$(v, w), (f_1, \dots, f_{|v|}, a_1, \dots, a_{|w|})\,\mathrm{real}^+\,\phi$$

and

$$(v, w), (f_1, \dots, f_{|v|}, a_1, \dots, a_{|w|})\,\mathrm{real}^-\,\phi$$

recursively as follows (where $s \in \{+, -\}$ and $Q \in \{\exists, \forall\}$):

(1) If ϕ is atomic, then

$$(v, w), (f_1, \dots, f_{|v|}, a_1, \dots, a_{|w|})\,\mathrm{real}^s\,\phi$$

is the conjunction of the formula expressing that ϕ has both free and bound second-order variables contained in $\{\alpha_1, \ldots, \alpha_{|v|}\}$ and first-order variables contained in $\{x_1, \ldots, x_{|w|}\}$, and of the formula which expresses that $\phi[\alpha_i := f_i, x_j := a_j]$ is true.

(2) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^s\ \phi \wedge \psi$$

if $(\phi \wedge \psi)[\alpha_i := f_i, x_j := a_j]$ is true,

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^s\ \phi$$

and

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^s\ \psi.$$

(3) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^s\ \phi \to \psi$$

if, $(\phi \to \psi)[\alpha_i := f_i, x_j := a_j]$ is true, and whenever

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^t\ \phi$$

we have

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^s\ \psi,$$

where t is the opposing sign of s.

(4) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \exists x_i \phi(x_i)$$

if $(\exists x_i \phi(x_i))[\alpha_p := f_p, x_q := a_q]$ is true,

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \phi(x_i),$$

and $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ is strongly valid for (v, w) at $w(i)$.

(5) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \forall x_i \phi(x_i)$$

if $(\forall x_i \phi(x_i))[\alpha_p := f_p, x_q := a_q]$ is true and

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \phi(x_i).$$

(6) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^-\ Q x_i \phi(x_i)$$

if $(Q x_i \phi(x_i))[\alpha_p := f_p, x_q := a_q]$ is true and

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^-\ \phi(x_i).$$

(7) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \exists \alpha_i \phi(\alpha_i)$$

if $(\exists \alpha_i \phi(\alpha_i))[\alpha_p := f_p, x_q := a_q]$ is true,

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \phi(\alpha_i),$$

and $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ is strongly valid for (v, w) at $v(i)$,

(8) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \forall \alpha_i \phi(\alpha_i)$$

if $(\forall \alpha_i \phi(\alpha_i))[\alpha_p := f_p, x_q := a_q]$ is true and

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^+\ \phi(\alpha_i).$$

(9) $$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^-\ Q \alpha_i \phi(\alpha_i)$$

if $(Q \alpha_i \phi(\alpha_i))[\alpha_p := f_p, x_q := a_q]$ is true and

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \text{ real}^-\ \phi(\alpha_i).$$

Definition 4.7. Given a sequent $\Gamma \vdash \phi$ we say that (v, w) is *monotone* for $\Gamma \vdash \phi$ if:

- For every subformula of Γ or ϕ of the form $Q x_i \psi$, if x_j is bound in ψ, then $w_1(i) < w_1(j)$, and if α_j is bound in ψ, then $w_1(i) < v_1(j)$.
- For every subformula of Γ or ϕ of the form $Q \alpha_i \psi$, if x_j is bound in ψ, then $v_1(i) < w_1(j)$, and if α_j is bound in ψ, then $v_1(i) < v_1(j)$.

Furthermore, we say that (v, w) is *universal* for $\Gamma \vdash \phi$ if for all x_i bound by a universal quantifier in a positive position (see Buss [4, 1.2.10]) or by an existential quantifier in a negative position we have that $w(i) \in \omega$, and similarly for such α_i that $v(i) \in \omega$.

We now let

$$(v, w) \text{ real } \Gamma \vdash \phi$$

be the formula expressing that v and w are monotone and universal for $\Gamma \vdash \phi$, and that for all sequences $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ which are valid for (v, w) we have that, if

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \operatorname{real}^- \psi$$

holds for all $\psi \in \Gamma$, then

$$(v, w), (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|}) \operatorname{real}^+ \phi$$

also holds.

Finally, if $\Gamma \vdash \phi$ is a Π^1_2-sequent, say

$$\Gamma = \forall\alpha_3\xi_1(\alpha_3) \to \exists\alpha_4\psi_1(\alpha_3, \alpha_4), \ldots, \forall\alpha_{2n+1}\xi_n(\alpha_{2n+1}) \to \exists\alpha_{2n+2}\psi_1(\alpha_{2n+1}, \alpha_{2n+2})$$

and $\phi = \forall\alpha_1\xi_0(\alpha_1) \to \exists\alpha_2\psi_0(\alpha_1, \alpha_2)$, then we say that (v, w) *strongly realises* $\Gamma \vdash \phi$ if, when we let m be maximal such that $v_1(2m) \leq v_1(2)$, then

- $v_1(3) < v_1(5) < \cdots < v_1(2m-1)$;
- $(v, w) \operatorname{real} \Gamma \vdash \phi$;
- $v_1(i) < w_1(j)$ whenever $w_1(j) > 0$ and $1 \leq i \leq 2m$;
- for all sequences $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ which are valid for (v, w) and for all $1 \leq i \leq m$ we have that, if $\xi_0(f_1)$ is true, and $\psi_j(f_{2j+1}, f_{2j+2})$ is true for every $1 \leq j < i$, then $\xi_i(f_{2i+1})$ is true, and $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ is strongly valid for (v, w) at $v(2i+1)$.
- for all sequences $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ which are valid for (v, w), if $\xi_0(f_1)$ is true, and $\psi_j(f_{2j+1}, f_{2j+2})$ is true for every $1 \leq j \leq n$, then $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ is strongly valid for (v, w) at $v(2)$.

Now that we have given the definition of our realisability notion, we will show that sequents which are provable in $\mathrm{EL}_0^{\mathrm{a}}$ have a realiser, and even provably so.

Theorem 4.8. *Let Σ be the set of formulas which are contraction-similar to an axiom of* EL_0. *Let $\Gamma \vdash \phi$ be a sequent provable in* $\mathrm{EL}_0^{\mathrm{a}}$ *from Σ by a free-cut free proof. Then there is a realiser (v, w) of $\Gamma \vdash \phi$, and in fact* RCA_0 *proves that* $(v, w) \operatorname{real} \Gamma \vdash \phi$.

The same holds if we replace EL_0 *by* $\mathrm{EL}_0 + \mathrm{MP}$ *and* $\mathrm{EL}_0^{\mathrm{a}}$ *by* $(\mathrm{EL}_0 + \mathrm{MP})^{\mathrm{a}}$

Proof. We prove this using induction on free-cut free proofs. In fact, we prove something slightly stronger: we even show that we can take $v(i) = 0$ for non-bound variables α_i and $v(j) = 0$ for non-bound variables x_j. Furthermore, we prove that we can assume that if some variable α_i is bound we have that $v_1(i) > 0$, and similarly if some variable x_j is bound then $w_1(j) > 0$. Finally, we will prove that we can take (v, w) to be self-contained.

(1) Identity (I) or equality axioms[4]: we can take v and w to be constantly 0 on a domain which is large enough to contain all variables.
(2) Weakening: we can obtain a realiser (v', w') for the lower sequent by extending the realiser (v, w) for the upper sequent to a big enough domain, and by choosing suitable values for $v'(i)$ and $w'(j)$ for bound variables occurring in the principal formula so as not to violate the extra conditions imposed in the induction claim; and letting $v'(i) = v(i)$ and $w'(j) = w(j)$ for other $i \leq |v|$ and $j \leq |w|$, and finally letting $v'(i) = 0$ and $w'(j) = 0$ for $i > |v|, j > |w|$ which do not represent any bound variables in the principal formula. It is not hard to see that this can be done and we omit the details.
(3) Permutation: the realiser for the upper sequent is also a realiser for the lower sequent.

[4]We assume the equality axioms are formulated without quantifiers, which we can do because they are universal.

(4) All non-quantifier logical rules with one upper sequent: we can use the same realiser for the lower sequent as for the upper sequent.

(5) All non-quantifier logical rules with two upper sequents: since no variable occurs twice, and since non-bound variables are mapped to 0, we can first restrict the realiser (v^1, w^1) that we get from the induction hypothesis for the left upper sequent to the variables occurring in the left upper sequent, and the realiser (v^2, w^2) for the right upper sequent to the variables occurring in the right upper sequent, and then their domains only contain free variables, on which they agree. Then as a realiser (v', w') for the lower sequent we can just take the union $(v^1 \cup v^2, w^1 \cup w^2)$ of these realisers with a restricted domain, where we add zeroes to extend their domains to an initial segment of ω.[5]

Here, let us note that we have a lot of extra freedom in choosing our realiser (v', w') of the lower sequent. For example, we could assume that all non-zero values of v_1^1, w_1^1 are strictly above the non-zero values of v_1^2, w_1^2 by scaling them before taking the union. We will use this fact below.

(6) $\exists x_u R$: let (v, w) realise the upper sequent. Now, let $i_1, \ldots, i_s$ list those $1 \leq i \leq |v|$ with $v(i) = 0$ in increasing order, and let $j_1, \ldots, j_t$ list those $1 \leq j \leq |w|$ with $w(j) = 0$ in increasing order. Note that for every free variable α_i we have $v(i) = 0$, and that for every free variable x_j we have $w(j) = 0$. So, let $t(\alpha_{i_1}, \ldots, \alpha_{i_s}, x_{j_1}, \ldots, x_{j_t})$ be the term which is captured by $\exists x_j R$. By Theorem 2.6 we can find an index e for the Turing functional mapping $(f_1, \ldots, f_s, a_1, \ldots, a_t) \in (\omega^\omega)^s \times \omega^t$ to $t(f_1, \ldots, f_s, a_1, \ldots, a_t)$. Now, given a realiser (v, w) for the upper sequent, we let $w'(u) = (1, e)$, we let $v_1'(i) = v_1(i) + 1$ if $v_1(i) \neq 0$ and we let $w_1'(j) = w_1(j) + 1$ if $w_1(j) \neq 0$ and $j \neq u$, and we let v' and w' agree with v and w otherwise.

(7) $\forall x_u L$: this step is analogous to $\exists x_u R$.

(8) $\forall x_u R$: let (v, w) realise the upper sequent. Now we let $w'(u) = 1$, we let $v_1'(i) = v_1(i) + 1$ if $v_1(i) \neq 0$ and we let $w_1'(j) = w_1(j) + 1$ if $w_1(j) \neq 0$ and $j \neq u$, and we let v' and w' agree with v and w otherwise. Then (v', w') realises the lower sequent.

(9) $\exists x_u L$: this step is analogous to $\forall x_u R$.

(10) $\exists \alpha_u R$: this step is analogous to $\exists x_u R$.

(11) $\forall \alpha_u L$: this step is analogous to $\exists \alpha_u R$.

(12) $\forall \alpha_u R$: this step is analogous to $\forall x_u R$.

(13) $\exists \alpha_u L$: this step is analogous to $\forall \alpha_u R$.

(14) An axiom which is an element of Σ: let us consider an instance of QF-AC^{0,0}; the other formulas are similar (cf. the induction steps for cuts below). Thus, consider the formula $\phi = \forall x_c \exists x_d \psi(x_c, x_d) \to \exists \alpha_i \forall x_b \psi(x_b, \alpha_i(x_b))$. Then we get a realiser (v, w) of $\vdash \phi$ by setting $w(c) = 3$, $w(d) = 4$, $w(i) = (1, e)$ and $w(b) = 2$, where e is an index for the computable function which, on input x_b, computes the least m such that $\psi(x_b, m)$ holds, which we can find because ψ is quantifier-free. Note that e is an index for a total computable function if $\forall x_c \exists x_d \psi(x_c, x_d)$ is true, which shows that (v, w) realises $\vdash \phi$.

(15) Cuts on a cut formula which is not contraction-similar to an instance of IA$_0$, QF-AC^{0,0} or MP: it can be directly verified that every realiser of the right upper sequent also realises the lower sequent, using the fact that all axioms which are not instances of IA$_0$ or QF-AC^{0,0} do not have any bound variables.

[5]Note that this does not work for the usual $\vee L$ rule for intuitionistic logic, since there is an implicit contraction hidden within this rule. However, our logic does not contain this rule because of our choice of interpreting $\phi \vee \psi$ as $\exists x (x = 0 \to \phi \wedge x \neq 0 \to \psi)$.

(16) Cuts on a cut formula which is contraction-similar to an instance of IA_0: let us first consider the case where the cut formula is actually an instance of IA_0. Thus, let us say that the cut formula is of the form

$$\phi = \forall x_i(\psi(x_i) \to \psi(S(x_i))) \to \forall x_j(\psi(0) \to \psi(x_j)),$$

where ψ is quantifier-free. Let (v, w) realise the right upper sequent, let $w'(i) = w'(j) = 0$, and let $v'(s) = v(s)$ and $w'(s) = w(s)$ otherwise.

We claim that (v', w') realises the lower sequent. In fact, we argue that (v, w) negatively realises ϕ, and after that we argue that this is enough to prove the claim. Let us reason within EL_0. Let $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ be valid for (v, w). Assume (v, w) positively realises $\forall x_i(\psi(x_i) \to \psi(S(x_i)))$ at the sequence $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$. In particular, it is true (at the valuation given by the sequence, which we will henceforth leave implicit). So, using induction we get that $\forall x_j(\psi(0) \to \psi(x_j))$ is true. In particular, $\psi_0 \to \psi(a_j)$ holds, and since ψ is quantifier-free this means that $\forall x_j(\psi_0 \to \psi(x_j))$ is negatively realised by (v, w) at $(f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$. So, (v, w) negatively realises ϕ.

Now, if $\sigma = (f_1, \ldots, f_{|v|}, a_1, \ldots, a_{|w|})$ is valid for (v', w'), then we can turn this into a sequence σ' which is valid for (v, w) by replacing a_j with the number computed from the other elements of σ using $\Phi_{w_2(j)}$ if $w(j) \in \omega \times \omega$ and this computation converges (here we use that (v, w) is self-contained). Let $\Delta \vdash \eta$ be the conclusion of the application of the cut rule. Then, if (v', w') negatively realises Δ at σ, so does (v, w) at σ' (again, using that (v, w) is self-contained), and we just argued that it negatively realises ϕ at σ' as well, so by our choice of (v, w) it positively realises η at σ'. We therefore see that (v', w') positively realises η at σ, as desired.

Now, if instead we only have that the cut formula is contraction-similar to an instance of IA_0, this means that at least one subformula χ has been replaced by $\chi \wedge \chi$. Let us assume there is exactly one such subformula; the general case then follows by induction. If χ is quantifier-free, we can use exactly the same proof as we gave in the previous paragraphs. If χ is an instance of IA_0, we get the result by applying the transformation we just described two times, since we can see $\chi \wedge \chi$ as two instances of IA_0. If $\chi = \forall x_i(\psi(x_i) \to \psi(S(x_i)))$, i.e. the cut formula is of the form

$$\begin{aligned}\phi =& (\forall x_i(\psi(x_i) \to \psi(S(x_i))) \wedge \forall x_{i'}(\psi(x_{i'}) \\ &\to \psi(S(x_{i'})))) \to \forall x_j(\psi(0) \to \psi(x_j)),\end{aligned}$$

the only changes we need to make is to additionally make $w'(i') = 0$. The case where $\chi = \forall x_j(\psi(0) \to \psi(x_j))$ is similar. This covers all the cases.

(17) Cuts on a cut formula which is contraction-similar an instance of $\mathrm{QF\text{-}AC}^{0,0}$: again, let us first say that the cut formula is actually an instance of $\mathrm{QF\text{-}AC}^{0,0}$, i.e. of the form

$$\phi = \forall x_c \exists x_d \psi(x_c, x_d) \to \exists \alpha_i \forall x_b \psi(x_b, \alpha_i(x_b)),$$

where ψ is quantifier-free; the general case follows as in case (16). Without loss of generality, the last step in the proof of the right upper sequent had ϕ as its principal formula, since we can always move the cuts upwards in the proof tree as much as possible. Furthermore, if ϕ was introduced in the right upper sequent through a weakening (i.e. ϕ was *only weakly introduced*, see [4, 2.4.4.2]), we can remove the weakening and the cut rule from the proof. Thus, we may assume we have a proof of the form

$$\dfrac{\phi \qquad \dfrac{\Delta \vdash \forall x_c \exists x_d \psi(x_c, x_d) \qquad \Delta', \exists \alpha_i \forall x_b \psi(x_b, \alpha_i(x_b)) \vdash \eta}{\Delta, \Delta', \phi \vdash \eta}}{\Delta, \Delta' \vdash \eta}$$

Let (v, w) be a realiser for $\Delta', \exists \alpha_i \forall x_b \psi(x_b, \alpha_i(x_b)) \vdash \eta$. Let e be an index for the functional which, given valuations for the free variables in $\psi(x_c, x_d)$ different from x_c and x_d, on input x_c, looks for the least value m such that $\psi(x_c, m)$ holds. We can do this because ψ is quantifier free. Now for every $j \neq i$, if $v(j)$ is of the form (k, e'), find a new index e'' which ignores the input α_i and replaces it by Φ_e; and do the same thing for $w(j)$ for every $j \neq b$. Finally, let $v'(i) = w'(b) = 0$.

We claim: (v', w') realises $\Delta, \Delta' \vdash \eta$. Let $(f_1, \ldots, f_{|v'|}, u_1, \ldots, u_{|w'|})$ be valid for (v', w'). Assume (v', w') negatively realises Δ and Δ' at this sequence. Then, in particular, all formulas in Δ are true, so $\forall x_c \exists x_d \psi(x_c, x_d)$ is true and therefore Φ_e is total. Furthermore, since it negatively realises Δ', it is now not hard to argue that, by replacing a_b in a similar way as in case (16), we have that (v, w) positively realises η, as desired.

(18) Cuts on a cut formula which is contraction-similar to an instance of MP: let us say that the cut formula is of the form $\phi = \neg\neg\exists x_b \psi(x_b) \to \exists x_c \psi(x_c)$, where ψ is quantifier-free; the general case follows in the same way as above. Again, without loss of generality, the last step in the proof of the right upper sequent had ϕ as its principal formula; thus, we have a proof of the form

$$\dfrac{\vdash \phi \qquad \dfrac{\Delta \vdash \neg\neg\exists x_b \psi(x_b) \qquad \Delta', \exists x_c \psi(x_c) \vdash \eta}{\Delta, \Delta', \phi \vdash \eta}}{\Delta, \Delta' \vdash \eta}$$

As before, let (v, w) realise $\Delta', \exists x_c \psi(x_c) \vdash \eta$. Let e be an index for the functional which computes the least number m such that $\psi(m)$ holds, which we can do because ψ is quantifier-free. Then, constructing (v', w') using a similar replacement strategy as in the previous case, we get a realiser (v', w') for $\Delta, \Delta' \vdash \psi$. □

Just having a realiser is, however, not enough to be able to extract a Weihrauch reduction. For this we need a strong realiser, and we will now prove these exist in a special case.

Definition 4.9. Let

$$\zeta_0 = \forall \alpha_0 \xi_0(\alpha_0) \to \exists \beta_0 \psi_0(\alpha_0, \beta_0), \ldots, \zeta_n = \forall \alpha_n \xi_n(\alpha_n) \to \exists \beta_n \psi_n(\alpha_n, \beta_n)$$

be Π^1_2-formulas such that every first-order quantifier in ξ_0 and in $\psi_1, \ldots, \psi_n$ is in the scope of a negation. Then we say that $\zeta_1, \ldots, \zeta_n \vdash \zeta_0$ is a *number-negative* Π^1_2-sequent.

Theorem 4.10. *Let Σ be the set of formulas which are contraction-similar to an axiom of* EL_0*. Let $\Gamma \vdash \forall \alpha_1 \xi_0(\alpha_1) \to \exists \alpha_2 \psi_0(\alpha_1, \alpha_2)$ be a number-negative Π^1_2 sequent provable in $\mathrm{EL}_0^{\mathrm{a}}$ from Σ with a free-cut free proof. Then there is a strong realiser (v, w) of $\Gamma' \vdash \phi$ for some permutation Γ' of a subsequence of Γ (and in fact RCA_0 proves that (v, w) real $\Gamma' \vdash \phi$).*

The same holds if we replace $\mathrm{EL}_0^{\mathrm{a}}$ by $(\mathrm{EL}_0 + \mathrm{MP})^{\mathrm{a}}$.

Proof. Let $\zeta_0 = \forall \alpha_1 \xi_0(\alpha_1) \to \exists \alpha_2 \psi_0(\alpha_1, \alpha_2)$. Fix a free-cut free proof from Σ of $\Gamma \vdash \zeta_0$ in $\mathrm{EL}_0^{\mathrm{a}}$. First, note that we may assume that all weakenings had as a principal formula a subformula of a formula of the form $\exists \alpha \eta$ where η does not contain any function quantifiers, as long as we move to a subsequence Γ' of Γ.

We now claim that the construction used in the proof of Theorem 4.8, applied to this proof, directly gives us a strong realiser (v, w) if we use the remark in case (5) in such a way that, whenever we combine two realisers (v^1, w^1) and (v^2, w^2), and one of them has $v_1^i(2) > 0$, then all non-zero values of v_1^{1-i}, w_1^{1-i} are above all non-zero values of v_1^i, w_1^i. Let us assume that Γ' consists of one formula $\zeta_1 = \forall\alpha_3\xi_1(\alpha_3) \to \exists\alpha_4\psi_1(\alpha_3, \alpha_4)$; the general case then follows by induction after we order the formulas in Γ' according to the values of v_1 at their variables.

Let us assume that $v_1(1) < v_1(3) < v_1(4) < v_1(2)$; the other cases are similar but easier. We need to show that $v_1(2) < w_1(i)$ for any $w_1(i) > 0$. There are four cases: x_i occurs in exactly one of ψ_0, ψ_1, ξ_0 or ξ_1. We will show that, if x_i occurs in ζ_1 or ζ_2, then it is bound during the proof at a stage[6] which is not above the stage at which α_2 is bound, which gives the result by closely inspecting the proof of Theorem 4.8. In fact, it is not hard to see that, if α_i and α_j are bound on the same path of the proof tree, then $v_1(i) < v_1(j)$ if and only if α_i is bound before α_j, and similarly for other combinations of function variables and number variables. In case x_i and α_2 are bound at incomparable stages s and s', then our choice of (v, w) directly ensures that $w_1(i) > v_1(2)$.

First, let us assume x_i appears in ψ_1. Towards a contradiction, let us assume that α_2 is already bound in some sequent s occurring in the proof, while x_i is bound somewhere below s. Then x_i eventually has to become bound, and since x_i is in the scope of a negation, there must be an application of $\neg L$ after the step in which this binding occurs and hence below s; say with conclusion $\Delta \vdash$. So $\exists\alpha_2\psi_0(\alpha_1, \alpha_2)$ is a subformula of a formula in Δ, and therefore we can never get to the conclusion $\Gamma \vdash \forall\alpha_1\xi_0(\alpha_1) \to \exists\alpha_2\phi_0(\alpha_1, \alpha_2)$ by the convention that every variable is bound at most once.

The case where x_i appears in ξ_0 is similar. Also, if x_i appears in ψ_0 then it is clearly bound before α_2. Finally, let us assume x_i appears in ξ_1. Then, by our assumption on the use of the weakening rule in our proof, we know that there is some application of $\to L$ of the form

$$\frac{\Delta \vdash \xi_1(\alpha_3) \qquad \Delta', \exists\alpha_4\psi_1(\alpha_3, \alpha_4) \vdash \eta}{\Delta, \Delta', \xi_1 \to \exists\alpha_4\psi_1(\alpha_3, \alpha_4) \vdash \eta}$$

Note that α_2 has to be bound in η, because α_4 is bound and otherwise we would have $v_1(2) < v_1(4)$, contradicting our assumption. Also note that x_i is bound in ξ_1. Therefore the stages at which x_i and α_2 are bound are incomparable, as desired. Let us also note that, using Lemma 4.11 below, we know that if ξ_0 is true, then Δ is true, and hence ξ_1 is true, which is one of the other things we needed to show to prove that (v, w) is a strong realiser.

Next, let us assume that $\xi_0(f_1)$ and $\psi_1(f_3, f_4)$ are true. Let $v(3) = (k, e)$. We claim: $\Phi_e(f_1)\downarrow$. Indeed, the only way this could be false is if, somewhere during the proof, we used cut elimination on a formula which is contraction-similar to an instance of QF-AC^{0,0}, after α_3 was bound. Let us consider an instance of QF-AC^{0,0}, say

$$\phi = \forall x_c \exists x_d \eta(x_c, x_d) \to \exists\alpha_i \forall x_b \eta(x_b, \alpha_i(x_b)),$$

the general case follows in the same way. By moving the cut up if necessary, the proof directly before the cut is of the following form.

$$\frac{\vdash \phi \qquad \dfrac{\Delta \vdash \forall x_c \exists x_d \psi(x_c, x_d) \qquad \Delta', \exists\alpha_i \forall x_b \psi(x_b, \alpha_i(x_b)) \vdash \eta}{\Delta, \Delta', \phi \vdash \eta}}{\Delta, \Delta' \vdash \eta}$$

[6]When we talk about the stage at which a variable is bound during the proof, we mean the derivation step in the proof in whose conclusion the variable first appears bound.

Since we know that α_3 was already bound at this stage of the proof, and we assumed that $v_1(3) < v_1(2)$, we know that α_2 was also already bound before this stage of the proof. Therefore, again using Lemma 4.11, we have that Δ follows from $\xi_0(f_1)$ and the axioms from Σ, and hence from $\xi_0(f_1)$. Thus, $\forall x_c \exists x_d \psi(x_c, x_d)$ is true, and therefore $\Phi_e(f_1)\downarrow$. By a similar argument, we see that $\Phi_e(f_1, f_3, f_4)\downarrow$, which completes the proof of the claim that (v, w) is a strong realiser of the sequent $\zeta_1 \vdash \zeta_0$, as desired.

Finally, if we add MP, it is generally no longer the case that, after x_i is bound, there needs to be an application of $\neg L$, as we used in the proof above. However, this can only happen if the stage at which x_i is bound is an instance of MP, but then there is nothing above this stage; in particular this cannot happen if α_2 is bound above this stage. So, the proof goes through directly. □

Lemma 4.11. *Assume that, in* $\mathrm{EL}_0 + \mathrm{MP}$ *there is a proof of* $\Delta, \Gamma \vdash \xi \to \exists \alpha \psi$ *from* $\Delta', \Gamma \vdash \exists \alpha \psi$*, and* s *is a valuation such that, whenever there is an application of* $\forall L \alpha_i$ *in this proof, which replaces the term* t *by* α_i*, we have that* $s(\alpha_i)$ *agrees with* $s(t)$*; and similarly for* $\forall L x_i$*. Then, if* Δ*,* ξ *and all cut formulas are true at* s*, so is* Δ'*.*

Proof. By induction on a free-cut free proof. For the induction step, we look at the topmost derivation rule. Note that the only rule which can be applied to the right-hand side is $\to R$ with $\xi \to \exists \alpha \psi$ as its principal formula, for which the proof is clear. The proof is also straightforward when the topmost rule only has one upper sequent. Next, consider the case where we have an application of $\to L$ of the following form.

$$\frac{\Delta_1 \vdash \eta_1 \qquad \Delta_2, \Gamma, \eta_2 \vdash \exists \alpha \psi}{\Delta_1, \Delta_2, \Gamma, \eta_1 \to \eta_2 \vdash \exists \alpha \psi}$$

By the induction hypothesis, if Δ and ξ are true, then so are Δ_1, Δ_2 and $\eta_1 \to \eta_2$. Then, because $\Delta_1 \vdash \eta_1$ is valid we also have that η_1 is true. Combining this with the fact that $\eta_1 \to \eta_2$ is true shows that η_2 is true, which shows that the statement of the lemma also holds for the upper right sequent.

Furthermore, in case the topmost rule is a cut, the induction step follows from the fact that all cut formulas are true. Finally, in case the topmost rule is $\forall L$, the induction step follows from the assumption on the valuation. □

5. From realisability to Weihrauch reducibility

In the previous section, we showed how to extract a realiser from a proof in EL_0. We now show how we can move from such realisers to Weihrauch reducibility.

Theorem 5.1. *If a* Π^1_2*-sequent* $\zeta_1, \ldots, \zeta_n \vdash \zeta_0$ *has a strong realiser, then there are natural numbers* $e_1, \ldots, e_{n+1}$ *such that* ζ_0 *Weihrauch-reduces to the composition of* $\zeta_n, \ldots, \zeta_1$*, as witnessed by* $e_1, \ldots, e_{n+1}$*. If* RCA_0 *proves the existence of this realiser, then it also proves that* ζ_0 *Weihrauch-reduces to the composition of* $\zeta_1, \ldots, \zeta_n$*.*

Proof. Let $\zeta_i = \forall \alpha_{2i+1} \xi_i(\alpha_{2i+1}) \to \exists \alpha_{2i+2} \psi_i(\alpha_{2i+1}, \alpha_{2i+2})$, and let (v, w) be a strong realiser. For ease of notation, let us assume that $|v| = 2n$, that $v_1(1) < v_1(3) < v_1(4) < \cdots < v_1(2n{+}1) < v_1(2n{+}2) < v_1(2)$, and that $v(3), v(5), \ldots, v(2n{+}1), v(2) \in \omega \times \omega$; the general case follows in the same way. We reason within RCA_0. Let f_1 be arbitrary such that $\xi_0(\alpha_{2i+1})$ holds. Next, let $f_3 = \Phi_{v_2(3)}(f_1)$ (which is total because (v, w) is a strong realiser), and let f_4 be arbitrary such that $\psi_1(f_3, f_4)$ holds. Continuing like this, for $1 < i < n$ we let $f_{2i+1} = \Phi_{v_2(2i+1)}(f_1, f_3, f_4, \ldots, f_{2i})$ and we let f_{2i+2} be arbitrary such that $\psi_i(f_{2i+1}, f_{2i+2})$ holds. Then $\xi_i(f_{2i+1})$ holds for every $1 \leq i \leq n$ because (v, w) is a strong realiser.

Now, let $f_2 = \Phi_{v_2(2)}(f_1, f_3, f_4, \dots, f_{2n+2})$, which is again total because (v, w) is a strong realiser. We claim: $(f_1, \dots, f_{2n+2})$ can be extended to a sequence which is valid for (v, w) and which negatively realises $\zeta_1, \dots, \zeta_n$ and ξ_0.

For ease of notation, let us assume $w_1(1) < w_1(2) < \cdots < w_1(|w|)$; otherwise we rename variables. Let us assume $a_1, \dots, a_{i-1}$ have been defined, and let us define a_i. If $w(i) = (k, e)$, let $a_i = \Phi_e(f_1, \dots, f_{2n+2}, a_1, \dots, a_{i-1})$ if this is defined, and 0 otherwise. If $w(i) \in \omega$, let η be the unique subformula of $\zeta_0, \zeta_1, \dots, \zeta_n$ such that η is of the form $Qx_i\eta'(x_i)$. If η is a subformula of ξ_j for some $j \geq 0$ or of ψ_j for some $j \geq 1$, call this formula of which η is a subformula ξ. Now remove from ξ all quantifiers of the form Qx_s for some $s < i$ to obtain ξ'. Next, choose a value a_i such that ξ' is true at $f_1, \dots, f_{2n+2}, a_1, \dots, a_i$. We can always do this because $\xi_0, \xi_1, \dots, \xi_n$ and $\psi_1, \dots, \psi_n$ are true, and because how we have inductively chosen $a_1, \dots, a_{i-1}$. On the other hand, if η is a subformula of ψ_0, let a_i be 0.

It is now not hard to check that $(f_1, \dots, f_{2n+2}, a_1, \dots, a_{|w|})$ is a valid sequence negatively realising $\zeta_1, \dots, \zeta_n$, and hence it positively realises ζ_0. Since it also negatively realises ξ_0, we see that it positively realises $\psi_0(f_1, f_2)$. In particular, $\psi_0(f_1, f_2)$ is true, which completes the proof that ζ_0 Weihrauch-reduces to the composition of $\zeta_1, \dots, \zeta_n$. □

We can now prove the first implication of our main theorem, by combining the theorems proven above.

Theorem 5.2. *If $\zeta_1 \to \zeta_0$ is a number-negative Π^1_2-sequent provable in* $\mathrm{EL}_0 + \mathrm{MP}$*, then there are natural numbers n and $e_1, \dots, e_{n+1}$ such that* RCA_0 *proves that ζ_0 Weichrauch-reduces to the composition of n copies of ζ_1, as witnessed by $e_1, \dots, e_{n+1}$.*

Proof. Let Σ be the set of formulas which are contraction-similar to an axiom of $\mathrm{EL}_0 + \mathrm{MP}$. From Lemma 4.5 we know that there is some sequent $\eta_1, \dots, \eta_n \vdash \psi$ which is contraction-similar to $\zeta_1 \vdash \zeta_0$ and which is provable in $(\mathrm{EL}_0 + \mathrm{MP})^{\mathrm{a}}$ from the axioms in Σ. Then, by Theorem 4.10 there is a strong realiser of $\Gamma \vdash \psi$, where Γ is a permutation of a subsequence of $\eta_n, \dots, \eta_1$. So, by Theorem 5.1 ζ_0 Weihrauch-reduces to the composition of Γ. All these steps are provable in RCA_0.

However, since every η_i is contraction-similar to ζ_1, we know from Lemma 4.4 that $\zeta_1 \leftrightarrow \eta_i$; similarly we have that $\zeta_0 \leftrightarrow \psi$. Thus, we have that ζ_0 is Weihrauch-reducible to the composition of finitely many copies of ζ_1, as desired. □

6. From Weihrauch reducibility to EL_0

We now prove the reverse implication of our main theorem. For this, we use the Kuroda negative translation.

Definition 6.1. (Kuroda [12]) The *Kuroda negative translation* of ϕ, ϕ^q, is defined as $\neg\neg\phi^*$, where ϕ^* is recursively defined by:

- $A^* = A$ for atomic formulas A.
- $(\phi \Box \psi)^* = \phi^* \Box \psi^*$ for $\Box \in \{\wedge, \to\}$.
- $(\exists x\phi)^* = \exists x\phi^*$ and $(\exists\alpha\phi)^* = \exists\alpha\phi^*$.
- $(\forall x\phi)^* = \forall x\neg\neg\phi^*$ and $(\forall\alpha\phi)^* = \forall\alpha\neg\neg\phi^*$.

Lemma 6.2. *If* $\mathrm{RCA}_0 \vdash \phi$*, then* $\mathrm{EL}_0 + \mathrm{MP} \vdash \phi^q$.

Proof. See e.g. Fujiwara [8, Lemma 11] or Kohlenbach [11, Propositions 10.3 and 10.6]; although they prove this result for RCA and EL + MP, it is easy to see that the same proof yields the desired result. □

Theorem 6.3. *Let $\zeta_i = \forall\alpha\xi_i(\alpha_i) \to \exists\beta_i\psi_i(\alpha_i, \beta_i)$ for $0 \leq i \leq n$ be Π^1_2-formulas. If there are natural numbers $e_1, \dots, e_{n+1}$ such that* RCA_0 *proves that ζ_0 Weihrauch-reduces to the composition of $\zeta_n, \dots, \zeta_1$ with reductions witnessed by $e_1, \dots, e_{n+1}$,* *and we let* ζ_i' *be the formula* ζ_i *with* ξ_i *replaced by* ξ_i^q *and* ψ_i *replaced by* ψ_i^q*, then* $\mathrm{EL}_0 + \mathrm{MP}$ *proves that* $(\zeta_1' \wedge \cdots \wedge \zeta_n') \to \zeta_0'$.

Proof. The fact that ζ_0 Weihrauch-reduces to the composition of $\zeta_1, \ldots, \zeta_n$, as witnessed by $e_1, \ldots, e_n + 1$, is expressed by a formula of the form

$$\forall \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n(\gamma(\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n)),$$

where γ is built using only the propositional connectives, from the formulas ξ_i and ψ_i, from formulas expressing that some functional is defined on given inputs, and formulas expressing that some α_i or β_i is equal to the output of some functional. Note that the latter two are Π_2^0 respectively Π_1^0. Also, note that for a Π_2^0-formula $\forall x \exists y \eta$, we have (in $\mathrm{EL}_0 + \mathrm{MP}$) that

$$(\forall x \exists y \eta)^q = \neg\neg\forall x \neg\neg\exists y \eta \Leftrightarrow \forall x \neg\neg\exists y \eta \Leftrightarrow \forall x \exists y \eta,$$

where the last equivalence follows by Markov's principle. Thus, we see (in $\mathrm{EL}_0 + \mathrm{MP}$) that

$$\begin{aligned}(\forall \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n(\gamma))^q &\Leftrightarrow \neg\neg\forall \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \neg\neg\gamma^* \\ &\Leftrightarrow \forall \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \neg\neg\gamma^* \\ &\Leftrightarrow \forall \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \gamma^q \\ &\Leftrightarrow \forall \alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \gamma',\end{aligned}$$

where γ' is the formula γ where every formula ξ_i is replaced by ξ_i^q and every formula ψ_i is replaced by ψ_i^q. Thus, using the previous lemma we see that $\mathrm{EL}_0 + \mathrm{MP}$ proves that ζ_0' Weihrauch-reduces to $\zeta_1', \ldots, \zeta_n'$. The result now easily follows. ☐

In particular, we can now prove our main result.

Theorem 6.4. *Let* $\zeta_i = \forall \alpha_i \xi_i(\alpha_i) \to \exists \beta_i \psi_i(\alpha_i, \beta_i)$ *for* $i \in \{0, 1\}$. *Then the following are equivalent:*

(1) *There are an* $n \in \omega$ *and* $e_1, \ldots, e_{n+1}$ *such that* RCA_0 *proves that* $e_1, \ldots, e_{n+1}$ *witness that* ζ_0 *Weihrauch-reduces to the composition of* n *copies of* ζ_1.
(2) *For* $\zeta_i' = \forall \alpha_i \xi_i^q(\alpha_i) \to \exists \beta_i \psi_i^q(\alpha_i, \beta_i)$ *we have that* $\mathrm{EL}_0 + \mathrm{MP}$ *proves that* $\zeta_1' \to \zeta_0'$.

Proof. From Theorem 5.2 and Theorem 6.3. ☐

Remark 6.5. Note that, if ξ_i and ψ_i are Π_2^0, then ζ_i' is equivalent to ζ_i over $\mathrm{EL}_0 + \mathrm{MP}$, as we saw in the proof of Theorem 6.3. Of course classically every Π_2^1-formula can be written in this form, but we cannot generally prove this over RCA_0.

So, if ζ_0 and ζ_1 happen to be in this special form, then we can eliminate the use of the Kuroda negative translation from the statement of the previous theorem. In particular, the implication from (2) to (1) holds in this restated way, even though ζ_0 and ζ_1 do not have to be number-negative. Intuitively, the reason this is not a problem is that we have made the Skolem functions explicit by reducing ζ_0 and ζ_1 to this simpler form, and therefore made the hidden computational content of the ξ_i and ψ_i accessible to our Weihrauch reduction.

7. True Weihrauch reducibility and affine logic

It turns out that, if we weaken our proof system for intuitionistic logic, we get a correspondence with *true* Weihrauch reducibility, i.e. without resorting to the composition of multiple copies of ζ_1. This gives us a proof-theoretic formulation of Weihrauch-reducibility in RCA_0.

Theorem 7.1. *Let* $\zeta_i = \forall \alpha_i \xi_i(\alpha_i) \to \exists \beta_i \psi_i(\alpha_i, \beta_i)$ *for* $i \in \{0,1\}$. *Then the following are equivalent:*

(1) There are $\mathrm{e}_1, \mathrm{e}_2$ *such that* RCA_0 *proves that* $\mathrm{e}_1, \mathrm{e}_2$ *witness that* ζ_0 *Weihrauch-reduces to* ζ_1.

(2) For $\zeta_i' = \forall \alpha_i \xi_i^q(\alpha_i) \to \exists \beta_i \psi_i^q(\alpha_i, \beta_i)$ *we have that* $(\mathrm{EL}_0 + \mathrm{MP})^{\exists \alpha \mathrm{a}}$ *proves that* $\zeta_1' \to \zeta_0'$.

Proof. First, the fact that (2) implies (1) follows directly from the proof of Theorem 5.2. Conversely, if (1) holds, then, as in the proof of Theorem 6.3, we know that $\mathrm{EL}_0 + \mathrm{MP}$ proves that ζ_0 Weihrauch-reduces to ζ_1. Since we can formulate this as the universal closure of an arithmetical formula η, we know that there is a free-cut free proof of η, and since η does not contain function quantifiers this proof is in fact in $(\mathrm{EL}_0 + \mathrm{MP})^{\exists \alpha \mathrm{a}}$.[7] Now it is not hard to verify that this proof can be extended to a proof of $\zeta_0 \to \zeta_1$ in $(\mathrm{EL}_0 + \mathrm{MP})^{\exists \alpha \mathrm{a}}$. □

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[7]There might be a weakening with as principal formula a formula of the form $\exists \alpha \forall z B(x, \alpha(z))$, which is later a subformula of a principal formula of a cut, but it is not hard to see that such a pair of a weakening and a cut can be eliminated.