MATH 571: MATHEMATICAL LOGIC
HOMEWORK SET 13, DUE AT 8:50 ON MONDAY, DEC. 7

BRING YOUR SOLUTIONS TO CLASS, OR SLIDE THEM UNDER THE DOOR OF VAN VLECK 403

1. Let $R \subseteq \mathbb{N}$ be the set of numbers a such that a is the Gödel number of a sentence. Argue that R is decidable, and hence representable in A_E.

2. *This long problem is worth twice as many points as the other problems.* Let $f : \mathbb{N}^k \to \mathbb{N}$ be a function. We say that f is *definable in* $\mathcal{N}$ if and only if there is a formula ρ with free variables $x_1, \dots, x_{k+1}$ such that, for all natural numbers $a_1, \dots, a_{k+1}$, we have that $\mathcal{N} \vDash \rho(a_1, \dots, a_{k+1})$ if and only if $f(a_1, \dots, a_k) = a_{k+1}$. For this problem, you are **not** allowed to use the theorem which says that decidable relations are definable in $\mathcal{N}$.
   a. Show that a function is definable in $\mathcal{N}$ if and only if its graph
   $$\{(a_1, \dots, a_{k+1}) \in \mathcal{N}^{k+1} \mid f(a_1, \dots, a_k) = a_{k+1}\}$$
   is definable in $\mathcal{N}$.
   b. Show that, for every natural number n, the constant function
   $$C_n(a_1, \dots, a_k) = n$$
   is definable in $\mathcal{N}$. Also, show that the successor function
   $$S(a_1) = a_1 + 1$$
   is definable in $\mathcal{N}$, and that for all natural numbers k and $i \leq k$ the projection function
   $$P_i^k(x_1, \dots, x_k) = x_i$$
   is definable in $\mathcal{N}$.
   c. Show that, if $f : \mathbb{N}^k \to \mathbb{N}$ and $g_1, \dots, g_k : \mathbb{N}^m \to \mathbb{N}$ are definable in $\mathcal{N}$, then the composition $f \circ (g_1, \dots, g_k)$, where
   $$(f \circ (g_1, \dots, g_k))(a_1, \dots, a_m) = f(g_1(a_1, \dots, a_m), \dots, g_k(a_1, \dots, a_m))$$
   is definable in $\mathcal{N}$.
   d. Assume that $g : \mathbb{N}^k \to \mathbb{N}$ and $h : \mathbb{N}^{k+2} \to \mathbb{N}$ are definable in $\mathcal{N}$. Let f be the function given recursively by
   $$f(0, a_2, \dots, a_{k+1}) = g(a_2, \dots, a_{k+1})$$
   $$f(a_1 + 1, a_2, \dots, a_{k+1}) = h(f(a_1, \dots, a_{k+1}), a_1, a_2, \dots, a_{k+1})$$
   (we say that f is defined from g and h by *primitive recursion*). Show that f is definable in $\mathcal{N}$. For this, you may use *without proof* that there is a formula β with three free variables x_1, x_2, x_3, which satisfies the property that for every finite sequence $(b_0, \dots, b_s)$ of natural numbers there exists a single natural number d such that for all $0 \leq i \leq s$ and all u we have that $\mathcal{N} \vDash \beta[\![d, i, u]\!]$ if and only if $b_i = u$ (we can build such a β using number-theoretic arguments).

(Hint: This is the hardest part of this problem. Write down a formula which says "there is a number d such that

- for all u, if $\beta[\![d, 0, u]\!]$ holds, then $g(a_2, \ldots, a_{k+1}) = u$, and
- for all $0 \leq i \leq a_1 - 1$, for all v, w, if $\beta[\![d, i, v]\!]$ and $\beta[\![d, i+1, w]\!]$ hold, then $h(v, i, a_2, \ldots, a_{k+1}) = w$, and
- for all z, if $\beta[\![d, a_1, z]\!]$ holds, then $a_{k+2} = z$.")

e. Let $g : \mathbb{N}^{k+1} \to \mathbb{N}$ be definable in $\mathcal{N}$. Also assume that, for all natural numbers $a_2, \ldots, a_{k+1}$, there exists a number a_1 such that $g(a_1, \ldots, a_{k+1}) = 0$. Let $f(a_2, \ldots, a_{k+1})$ be the least number a_1 such that $g(a_1, \ldots, a_{k+1}) = 0$ (we say that f is defined by *minimization*. Show that f is definable in $\mathcal{N}$.

Remark: the functions which can be formed using b-e are called the μ-recursive functions. We have just shown that the μ-recursive functions are definable in $\mathcal{N}$. It is known that the computable functions are exactly the same as the μ-recursive functions, and hence this gives an alternative proof of the fact that all computable functions are definable in $\mathcal{N}$.

3. A theory T (in a language with 0 and S) is called ω-complete if and only if for any formula φ and variable x, if $\varphi[x := S^n(0)]$ belongs to T for every natural number n, then $\forall x \varphi$ belongs to T. Show that if T is a consistent ω-complete theory in the language of $\mathcal{N}$ and if $A_E \subseteq T$, then $T = \mathrm{Th}(\mathcal{N})$. (Suggestion: We need to show that, for every sentence σ, we have that $\sigma \in T$ if and only if $\mathcal{N} \vDash \sigma$. Write σ in prenex normal form, and use induction on the number of quantifiers.)