MATH 571: MATHEMATICAL LOGIC
HOMEWORK SET 2, DUE AT 8:50 ON FRIDAY, SEPT. 18

BRING YOUR SOLUTIONS TO CLASS, OR SLIDE THEM UNDER THE DOOR OF VAN VLECK 403

1. Exercise 1.2.1 from Enderton.

2. Is $(((P \to Q) \to P) \to P)$ a tautology? If your answer is yes, provide a truth table. If your answer is no, give a valuation which refutes the formula.

3. Exercise 1.2.3 from Enderton. If your answer is yes, provide a truth table. If your answer is no, give a valuation which refutes the formula.

4. Exercise 1.2.4 from Enderton. (Hint: Induction is not necessary here. You can prove this directly from the definitions of $\vDash$, $\to$, and $\leftrightarrow$ by looking at all possible valuations.)

5. For a wff φ, let $\mathrm{Var}(\varphi)$ be the set of sentence symbols which occur in φ. If φ is a wff such that $\mathrm{Var}(\varphi) \subseteq \{A_0, \ldots, A_n\}$, and $\psi_0, \ldots, \psi_n$ are wffs, then we write $\varphi(\psi_0, \ldots, \psi_n)$ for the formula where every occurrence of the sentence symbol A_i in φ is replaced by ψ_i. For example, if φ is the formula $(A_0 \vee A_1)$, then $\varphi(\psi_0, \psi_1)$ is the formula $(\psi_0 \vee \psi_1)$.
   (a) Show that $\varphi(\psi_0, \ldots, \psi_n)$ is again a wff (Hint: apply the induction principle to the set

$$S = \{\varphi \text{ wff} \mid \text{for all } n \in \mathbb{N}, \text{ if } \mathrm{Var}(\varphi) \subseteq \{A_0, \ldots, A_n\}, \text{ then for all wff } \psi_0, \ldots, \psi_n \text{ we have that } \varphi(\psi_0, \ldots, \psi_n) \text{ is a wff}\}.)$$

   (b) Let v be a truth assignment for the set of all sentence symbols. Define a truth assignment u by for the set $\{A_0, \ldots, A_n\}$ by $u(A_i) = \overline{v}(\psi_i)$. Show that

$$\overline{u}(\varphi) = \overline{v}(\varphi(\psi_0, \ldots, \psi_n)).$$

   (Hint: use induction.)
   (c) Show that, if φ is a tautology, then for all wff $\psi_0, \ldots, \psi_n$, we have that $\varphi(\psi_0, \ldots, \psi_n)$ is again a tautology. We therefore say that the set of tautologies is *closed under substitution.* (Hint: Induction is not necessary here. Use part (b).)