MATH 571: MATHEMATICAL LOGIC
HOMEWORK SET 3, DUE AT 8:50 ON FRIDAY, SEPT. 25

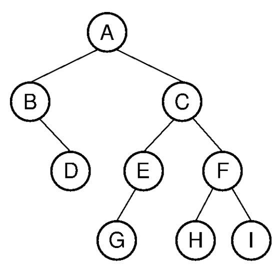
BRING YOUR SOLUTIONS TO CLASS, OR SLIDE THEM UNDER THE DOOR OF VAN VLECK 403

1. Exercise 1.4.2 from Enderton.

2. Let $U = \mathbb{N}$. For each of the following problems, give sets B and $\mathcal{F}$ consisting of exactly one element each such that the set C generated from B by $\mathcal{F}$ is:
   (a) The set of even numbers.
   (b) The set of odd numbers.
   (c) The set of squares, i.e. numbers $n \in \mathbb{N}$ such that $n = m^2$ for some $m \in \mathbb{N}$.

3. Define, by recursion on the set of well-formed formulas, the function $\overline{h}$ such that:
   i. $\overline{h}(A_i) = 0$ for every sentence symbol A_i.
   ii. $\overline{h}((\varphi \wedge \psi)) = \max(\overline{h}(\varphi), \overline{h}(\psi)) + 1$.
   iii. $\overline{h}((\varphi \vee \psi)) = \max(\overline{h}(\varphi), \overline{h}(\psi)) + 1$.
   iv. $\overline{h}((\varphi \to \psi)) = \max(\overline{h}(\varphi), \overline{h}(\psi)) + 1$.
   v. $\overline{h}((\varphi \leftrightarrow \psi)) = \max(\overline{h}(\varphi), \overline{h}(\psi)) + 1$.
   vi. $\overline{h}((\neg\varphi)) = \overline{h}(\varphi) + 1$.

   The value $\overline{h}(\varphi)$ is called the *depth* of φ. The *height* of a tree is defined as the number of edges on a maximal path between the root and a leaf; that is, the maximum number of steps one can go down from the top node. For example, the height of the following tree is 3:



   Show that the depth of a formula is equal to the height of its ancestral tree.

4. Let $\alpha : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection (it is known that such bijections exist, you do not need to prove this). Let $U = \mathbb{N}$, let
$$B = \{\alpha(0, n) \mid n \in \mathbb{N}\},$$
and let
$$\mathcal{F} = \{f\},$$
where
$$f(n, m) = \alpha(1, \alpha(n, m)).$$
Let C be the set generated from B by $\mathcal{F}$.

(a) Show that C is freely generated from B by $\mathcal{F}$.

(b) Let V be the set of all nonempty finite sequences of natural numbers. There is a natural operation on the set of finite sequences called *concatenation*, denoted by $u^\frown v$: the concatenation of two finite sequences u and v is the finite sequence obtained by putting the elements from the sequence v after the elements of the sequence u. For example, the concatenation of the sequences $(5, 17, 21)$ and $(1, 3)$ is $(5, 17, 21, 1, 3)$.
Show that there is a surjection β from C onto V such that $\beta(f(n, m)) = \beta(n)^\frown\beta(m)$.