MATH 571: MATHEMATICAL LOGIC
HOMEWORK SET 9, DUE AT 8:50 ON FRIDAY, NOV. 6

BRING YOUR SOLUTIONS TO CLASS, OR SLIDE THEM UNDER THE DOOR OF VAN VLECK 403

1. Let $\mathcal{L}$ be the language only containing equality. Show that there is no formula φ such that:
   $\mathcal{A} \vDash \varphi$ if and only $|\mathcal{A}|$ is either infinite, or is finite and contains an even number of elements. (Hint: assume such a formula φ exists, consider $\neg\varphi$, and derive a contradiction using one of the theorems discussed on Friday.)

2. Problem 2.5.8 from Enderton.

3. Let $\mathcal{L}$ be the first-order language containing one binary relation $<$ and equality. A *linear ordering* is a binary relation $<$ satisfying:
   - For every x, not $x < x$.
   - For no x, we have both $x < y$ and $y < x$.
   - For every x, y, z, if $x < y$ and $y < z$, then $x < z$.
   - If $x \neq y$, then $x < y$ or $y < x$.

   A linear order is a *well-ordering* if every non-empty subset B has a least element, i.e. there exists an element $b \in B$ such that for no $x \in B$ we have $x < b$.

   a. Give a formula φ such that $\mathcal{A} \vDash \varphi$ if and only if $<^{\mathcal{A}}$ is a linear ordering.

   b. Show that there is no set Σ such that $\mathcal{A} \vDash \Sigma$ if and only if $<^{\mathcal{A}}$ is a well-ordering. (Hint: Suppose that such a set Σ exists. Let $\mathcal{L}'$ be the language obtained from $\mathcal{L}$ by adding infinitely many new constants $c_1, c_2, \ldots$. Now let Σ' be the set Σ together with formulas of the form $c_i > c_j$ for $j > i$, and apply the compactness theorem to get a contradiction.)