MATH 571: LECTURE NOTES

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1. Introduction

As discussed during class, there are three main types of proof systems:

- Hilbert Calculus
- Natural Deduction
- Sequent Calculus.

Enderton discusses the first, while during this course we will mostly be discussing natural deduction. In these lecture notes I will describe the main points discussed in class, but you are advised to take your own notes during class as well, since these lecture notes will not be comprehensive. These lecture notes will be updated throughout the semester and are a replacement for sections 2.4 and 2.5 of Enderton.

2. Proof trees

The idea of proof systems is that we want an easier way of showing that some formula is a tautology, or more generally, that some set Σ of formulas logically implies some other formula φ. We do this using *proof trees*, which can be seen as a formalized version of the proofs studied in mathematics.

Our proofs in natural deduction will have the form of trees. We will work from assumptions at the top, and work towards conclusions towards the bottom. During the course of the proof, some of the rules that allow us to form proofs may allow us to *cross out* assumptions. In the end, if all of the assumptions which have not been crossed out are in Σ, and φ is the formula at the bottom of the tree, we will say that this is a proof that Σ implies φ (the formal terminology will follow below).

We will build our proofs using *rules*, which will allow us to form bigger proof trees from smaller proof trees. These rules are as follows. As discussed in class, $\varphi[x := t]$ denotes the formula where every free occurrence of the variable x is replaced by the term t; that is, we replace x by t unless there is a quantifier which says something about the variable x.

Every connective and quantifier will have two kinds of rules: *introduction rules*, which allow us to get a conclusion including this connective or quantifier, and *elimination rules*, which allow us to reason from a formula which contains the connective or quantifier.

Date: October 30, 2015.

$$(\wedge I)\ \frac{\varphi \qquad \psi}{\varphi \wedge \psi} \qquad\qquad (\wedge E1)\ \frac{\varphi \wedge \psi}{\varphi}$$

$$(\vee I1)\ \frac{\varphi}{\varphi \vee \psi} \qquad\qquad (\wedge E2)\ \frac{\varphi \wedge \psi}{\psi}$$

$$(\vee I2)\ \frac{\psi}{\varphi \vee \psi} \qquad\qquad (\vee E)\ \frac{\varphi \vee \psi \qquad \varphi \to \alpha \qquad \psi \to \alpha}{\alpha}$$

$$(\to I)\ \frac{\psi}{\varphi \to \psi} \qquad\qquad (\to E)\ \frac{\varphi \to \psi \qquad \varphi}{\psi}$$

where we are allowed to cross out any φ occurring as an assumption in the proof of ψ.

$$(\neg I)\ \frac{\varphi \to \psi \qquad \varphi \to \neg\psi}{\neg\varphi} \qquad\qquad (\neg E)\ \frac{\neg\neg\varphi}{\varphi}$$

$$(\exists I)\ \frac{\varphi[x := t]}{\exists x\varphi} \qquad\qquad (\exists E)\ \frac{\exists x\varphi \qquad \forall y(\varphi[x := y] \to \psi)}{\psi}$$

where y is not a free variable in ψ.

$$(\forall I)\ \frac{\varphi[x := y]}{\forall x\varphi} \qquad\qquad (\forall E)\ \frac{\forall x\varphi}{\varphi[x := t]}$$

where y is not a free variable in any of the assumptions in the proof of φ which are not yet crossed out.

Finally, if we are looking at a first-order language $\mathcal{L}$ containing equality, then we additionally have the following rules.

$$(R)\ \frac{}{\forall x(x = x)}$$

$$(S)\ \frac{}{\forall x\forall y(x = y \to y = x)}$$

$$(T)\ \frac{}{\forall x\forall y\forall z(x = y \wedge y = z \to x = z)}$$

$$(\mathrm{Sub}_f)\ \frac{}{\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n(x_1 = y_1 \wedge \cdots \wedge x_n = y_n \to f(x_1, \dots, x_n) = f(y_1, \dots, y_n))}$$

where f is an n-ary function in $\mathcal{L}$.

$$(\mathrm{Sub}_R)\ \frac{}{\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n(x_1 = y_1 \wedge \cdots \wedge x_n = y_n \wedge R(x_1, \dots, x_n) \to R(y_1, \dots, y_n))}$$

where R is an n-ary relation in $\mathcal{L}$.

Now, let us make the following definition.

Definition 2.1. Let Σ be a set of formulas and let φ be a formula. Then we say that Σ *derives* φ, or $\Sigma \vdash \varphi$, if and only if there is a proof tree, built using the rules described above, such that φ is at the bottom of the tree, and all assumptions which are not crossed out are elements of Σ.

We say that φ is *derivable*, or $\vdash \varphi$, if and only if $\emptyset \vdash \varphi$.

Let us give some examples of proofs; more examples were given in class.

Example 2.2. The following is a proof tree which shows that $\vdash (\varphi \wedge \psi) \to (\psi \wedge \varphi)$.

$$
\begin{prooftree}
\AxiomC{$\cancel{\varphi \wedge \psi}^{(1)}$}
\LeftLabel{$(\wedge E2)$}
\UnaryInfC{ψ}
\AxiomC{$\cancel{\varphi \wedge \psi}^{(1)}$}
\LeftLabel{$(\wedge E1)$}
\UnaryInfC{φ}
\LeftLabel{$(\wedge I)$}
\BinaryInfC{$\psi \wedge \varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(1)}
\UnaryInfC{$(\varphi \wedge \psi) \to (\psi \wedge \varphi)$}
\end{prooftree}
$$

Example 2.3. The following is a (hard) proof tree which shows that $\vdash \varphi \vee \neg\varphi$.

$$
\begin{prooftree}
\AxiomC{$\cancel{\neg(\varphi \vee \neg\varphi)}^{(3)}$}
\LeftLabel{$(\to I)$}
\RightLabel{(3)}
\UnaryInfC{$\neg(\varphi \vee \neg\varphi) \to \neg(\varphi \vee \neg\varphi)$}
\AxiomC{$\cancel{\neg(\varphi \vee \neg\varphi)}^{(2)}$}
\LeftLabel{$(\to I)$}
\UnaryInfC{$\varphi \to \neg(\varphi \vee \neg\varphi)$}
\AxiomC{$\cancel{\varphi}^{(1)}$}
\LeftLabel{$(\vee I1)$}
\UnaryInfC{$\varphi \vee \neg\varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(1)}
\UnaryInfC{$\varphi \to (\varphi \vee \neg\varphi)$}
\LeftLabel{$(\neg I)$}
\BinaryInfC{$\neg\varphi$}
\LeftLabel{$(\vee I2)$}
\UnaryInfC{$\varphi \vee \neg\varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(2)}
\UnaryInfC{$\neg(\varphi \vee \neg\varphi) \to (\varphi \vee \neg\varphi)$}
\LeftLabel{$(\neg I)$}
\BinaryInfC{$\neg\neg(\varphi \vee \neg\varphi)$}
\LeftLabel{$(\neg E)$}
\UnaryInfC{$\varphi \vee \neg\varphi$}
\end{prooftree}
$$

Example 2.4. The following is a proof tree which shows that $\vdash \neg\exists x\varphi \to \forall x\neg\varphi$.

$$
\begin{prooftree}
\AxiomC{$\cancel{\varphi}^{(1)}$}
\LeftLabel{$(\exists I)$}
\UnaryInfC{$\exists x\varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(1)}
\UnaryInfC{$\varphi \to \exists x\varphi$}
\AxiomC{$\neg\exists x\varphi$}
\LeftLabel{$\to I$}
\UnaryInfC{$\varphi \to \neg\exists x\varphi$}
\LeftLabel{$(\neg I)$}
\BinaryInfC{$\neg\varphi$}
\LeftLabel{$(\forall I)$}
\UnaryInfC{$\forall x\neg\varphi$}
\end{prooftree}
$$

Example 2.5. The following is a proof tree which shows that $\vdash \forall x\neg\varphi \to \neg\exists x\varphi$.

$$
\begin{prooftree}
\AxiomC{$\cancel{\exists x\varphi}^{(2)}$}
\AxiomC{$\cancel{\forall x\neg\varphi}^{(4)}$}
\LeftLabel{$(\forall E)$}
\UnaryInfC{$\neg\varphi$}
\LeftLabel{$(\to I)$}
\UnaryInfC{$(\exists x\varphi) \to \neg\varphi$}
\AxiomC{$\cancel{\varphi}^{(1)}$}
\LeftLabel{$(\to I)$}
\UnaryInfC{$(\exists x\varphi) \to \varphi$}
\LeftLabel{$(\neg I)$}
\BinaryInfC{$\neg\exists x\varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(1)}
\UnaryInfC{$\varphi \to \neg\exists x\varphi$}
\LeftLabel{$(\forall I)$}
\UnaryInfC{$\forall y(\varphi[x := y] \to \neg\exists x\varphi)$}
\LeftLabel{$(\exists E)$}
\BinaryInfC{$\neg\exists x\varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(2)}
\UnaryInfC{$\exists x\varphi \to \neg\exists x\varphi$}
\AxiomC{$\cancel{\exists x\varphi}^{(3)}$}
\LeftLabel{$(\to I)$}
\RightLabel{(3)}
\UnaryInfC{$\exists x\varphi \to \exists x\varphi$}
\LeftLabel{$(\neg I)$}
\BinaryInfC{$\neg\exists x\varphi$}
\LeftLabel{$(\to I)$}
\RightLabel{(4)}
\UnaryInfC{$\forall x\neg\varphi \to \neg\exists x\varphi$}
\end{prooftree}
$$

The following theorems, which we proved during class, tell us there is a tight connection between $\vDash$ and $\vdash$.

Theorem 2.6. (Soundness theorem) *Let Σ be a set of sentences and let φ be a sentence. If $\Sigma \vdash \varphi$, then $\Sigma \vDash \varphi$.*

Definition 2.7. We say that a set Σ is *consistent* if for no formula ψ we have $\Sigma \vdash \psi \wedge \neg\psi$.

Theorem 2.8. (Completeness theorem, formulation 1) *If Σ is a consistent set of sentences, then Σ is satisfiable, i.e. there is a model $\mathcal{A}$ such that $\mathcal{A} \vDash \Sigma$.*

Corollary 2.9. (Completeness theorem, formulation 2) *Let Σ be a set of sentences and let φ be a sentence. If $\Sigma \vDash \varphi$, then $\Sigma \vdash \varphi$.*

Proof. Towards a contradiction, assume $\Sigma \nvdash \varphi$. Then $\Sigma \cup \{\neg\varphi\}$ is consistent, because if $\Sigma \cup \{\neg\varphi\} \vdash \psi \wedge \neg\psi$, then we can extend this to a proof of $\Sigma \vdash \varphi$. Thus, by the previous theorem, there is a model $\mathcal{A}$ such that $\mathcal{A} \vDash \Sigma$ and $\mathcal{A} \vDash \neg\varphi$, so $\mathcal{A} \nvDash \varphi$. So $\Sigma \nvDash \varphi$. □

We can now also easily prove the compactness theorem for first-order logic.

Theorem 2.10. (Compactness theorem) *If Σ is a set of sentences such that every finite subset of Σ is satisfiable, then Σ is satisfiable.*

Proof. Let Σ be such that every finite subset of Σ is satisfiable. Towards a contradiction, assume Σ is not satisfiable. Let ψ be an arbitrary fixed sentence. Then $\Sigma \vDash \psi \wedge \neg\psi$, since there is no model of Σ and therefore trivially every $\mathcal{A}$ such that $\mathcal{A} \vDash \Sigma$ satisfies $\mathcal{A} \vDash \psi \wedge \neg\psi$.

Now, by the completeness theorem, we know that $\Sigma \vdash \psi \wedge \neg\psi$. Fix a proof P with conclusion $\psi \wedge \neg\psi$ and open assumptions among Σ. Since a proof is a finite object, P has only finitely many open assumptions; thus, if we let Γ be the set of open assumptions of P, this is a finite subset of Σ. Also, the proof P shows that $\Gamma \vdash \psi \wedge \neg\psi$, so by the soundness theorem we see that $\Gamma \vDash \psi \wedge \neg\psi$.

Since Γ is a finite subset of Σ, we know that there is a model $\mathcal{B}$ such that $\mathcal{B} \vDash \Gamma$. But then also $\mathcal{B} \vDash \psi \wedge \neg\psi$ since $\Gamma \vDash \psi \wedge \neg\psi$. But then both $\mathcal{B} \vDash \psi$ and $\mathcal{B} \nvDash \psi$, a contradiction.

So, our assumption was false and Σ is satisfiable. □