The Medvedev Lattice of Degrees of Difficulty

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1 Introduction

The Medvedev lattice was introduced in [5] as an attempt to make precise the idea, due to Kolmogorov, of identifying true propositional formulas with identically "solvable" problems. A *mass problem* is any set of functions (throughout this paper "function" means total function from ω to ω; the small Latin letters $f, g, h, \ldots$ will be used as variables for functions). Mass problems correspond to informal problems in the following sense: given any "informal problem", a mass problem corresponding to it is a set of functions which "solve" the problem, and at least one such function can be "obtained" by any "solution" to the problem (see [10]).

Example 1.1 If $A, B \subseteq \omega$ are sets, and ϕ is a partial function, then the following are mass problems:

1. $\{c_A\}$ (where c_A is the characteristic function of A): this is called the *problem of solvability of* A; this mass problem will be denoted by the symbol $\mathcal{S}_A$;

2. $\{f : \text{range}(f) = A\}$: the *problem of enumerability of* A; this mass problem will be denoted by the symbol $\mathcal{E}_A$;

3. (Other examples) The *problem of separability of* A *and* B, i.e. $\{f : f^{-1}(0) = A \ \& \ f^{-1}(1) = B\}$; of course, this mass problem is empty if $A \cap B \neq \emptyset$: it is absolutely impossible to "solve" the problem in this case. The *problem of many-one reducibility of* A *to* B: $\{f : f^{-1}(B) = A\}$. The *problem of extendibility of* ϕ: $\{f : f \supseteq \phi\}$.

*Research partially supported by Human Capital and Mobility network *Complexity, Logic and Recursion Theory*

A mass problem is *solvable* if it contains a recursive function, i.e., informally, if we have some ways of effectively computing a solution to the corresponding informal problem. Corresponding to this definition of solvable mass problem, there is an intuitive notion of reducibility between mass problems. If $\mathcal{A}$ and $\mathcal{B}$ are mass problems, we say that $\mathcal{A}$ is (informally) *reducible* to $\mathcal{B}$ if there is an effective procedure which, given any member of $\mathcal{B}$, yields a member of $\mathcal{A}$, i.e. we have an effective procedure for solving the problem corresponding to $\mathcal{A}$ given any solution to the problem corresponding to $\mathcal{B}$. The degree structure originated by this notion of reducibility turns out to be a distributive lattice (in fact a Brouwer algebra), called the *Medvedev lattice*. Despite its richness and, at the same time, the pleasant regularity of its properties, the Medvedev lattice has not been extensively studied. Rogers ([10]) raised several questions about this lattice: many of these questions are still open. The Medvedev lattice provides a more extensive context for both Turing reducibility and enumeration reducibility. It is to be expected that many of the global questions concerning these reducibilities can be profitably investigated in this wider context. There are several problems concerning automorphisms, automorphism bases and definability. Very little is known about filters, ideals and congruences. There is a variety of questions related to the Brouwer algebra structure of the Medvedev lattice; some of these questions deal with the problem of embedding Brouwer algebras in the Medvedev lattice and its initial segments, with consequent applications to intermediate propositional logics.

In this paper we review some of the existing literature on the Medvedev lattice. During the exposition, we list several open problems.

1.1 The formal definition of reducibility

The formal definition of reducibility between mass problems is given through the notion of a recursive operator. We shall refer to a listing $\{\Psi_z : z \in \omega\}$ of a large enough set of recursive operators, i.e. such that

1. $\Psi(\phi)$ is defined (i.e. a partial function), for all partial functions ϕ;

2. $(\forall \text{ recursive operator } \Omega)(\exists z)(\forall f)[\Psi_z(f) = \Omega(f)]$:

see [10] for a proof that such a listing exists.

Definition 1.2 We say that the mass problem $\mathcal{A}$ is *reducible* to the mass problem $\mathcal{B}$ (notation: $\mathcal{A} \leq \mathcal{B}$) if $(\exists z)[\Psi_z(\mathcal{B}) \subseteq \mathcal{A}]$. (Notice that $\Psi_z(\mathcal{B}) \subseteq \mathcal{A}$ implies that $\Psi_z(f)$ is total, for every $f \in \mathcal{B}$.)

The equivalence class $[\mathcal{A}]$ of $\mathcal{A}$ under the equivalence relation $\equiv$ generated by $\leq$ is called the *degree of difficulty* of $\mathcal{A}$. Let M be the set of degrees of difficulty: M is partially ordered by the relation $\leq$ defined by $[\mathcal{A}] \leq [\mathcal{B}]$ if and only if $\mathcal{A} \leq \mathcal{B}$. In fact,

Theorem 1.3 ([5]) *The structure* $\mathfrak{M} = \langle M, \leq \rangle$ *is a distributive lattice with* $0, 1$.

Proof. We just recall how the lattice–theoretic operations are defined. Given $n \in \omega$, functions f, g, and mass problems $\mathcal{A}$, $\mathcal{B}$, let $n * f$ denote the function defined by the clauses: $n * f(0) = n$, and $n * f(x) = f(x-1)$, if $x > 0$; let $f \vee g$ denote the function defined by the following clauses: $f \vee g\,(2x) = f(x)$, and $f \vee g\,(2x+1) = g(x)$; finally let $n * \mathcal{A} = \{n * f : f \in \mathcal{A}\}$, and $\mathcal{A} \vee \mathcal{B} = \{f \vee g : f \in \mathcal{A}\ \&\ g \in \mathcal{B}\}$. Let now $\mathcal{A}$, $\mathcal{B}$ be mass problems: define $[\mathcal{A}] \wedge [\mathcal{B}] = [0 * \mathcal{A} \cup 1 * \mathcal{B}]$; $[\mathcal{A}] \vee [\mathcal{B}] = [\mathcal{A} \vee \mathcal{B}]$. Finally, let $\mathbf{0} = [\mathcal{A}]$, where $\mathcal{A}$ contains at least a recursive function, and $\mathbf{1} = [\emptyset]$.

It is easy to see that the above are well given definitions and make $\mathfrak{M}$ a distributive lattice, $\mathbf{0}$ being the least element, and $\mathbf{1}$ being the greatest element. □

Notice that $\mathbf{0}$ is the "easiest" degree of difficulty, containing only solvable problems; $\mathbf{1}$ is the "most difficult" degree of difficulty: it is absolutely impossible to solve the empty mass problem!

In the following, if P is any property of mass problems, we say that a degree of difficulty $\mathbf{A}$ has the property P if $\mathbf{A}$ contains some mass problem $\mathcal{A}$ having the property P. Thus, for instance $\mathbf{A}$ is *discrete*, if $\mathbf{A}$ contains some discrete mass problem in the Baire topology of ω^ω, etc.

2 A common framework for Turing degrees and enumeration degrees

Definition 2.1 For every set A, let $\mathbf{S}_A = [\mathcal{S}_A]$, and $\mathbf{E}_A = [\mathcal{E}_A]$. The degree $\mathbf{S}_A$ is called the *degree of solvability of* A; $\mathbf{E}_A$ is called the *degree of enumerability of* A.

Theorem 2.2 ([5]) 1. *The mapping* $[A]_T \mapsto \mathbf{S}_A$ *is an embedding of the structure* $\mathfrak{D}_T$ *of the Turing degrees onto the degrees of solvability, preserving* $0, \vee$.

2. *The mapping* $[A]_e \mapsto \mathbf{E}_A$ *is an embedding of the structure* $\mathfrak{D}_e$ *of the enumeration degrees onto the degrees of enumerability, preserving* $0, \vee$.

Proof. The proof of 1. is immediate. As to 2, it is enough to observe that if $A \leq_e B$ then there is an effective procedure for enumerating A given any enumeration of B; use this procedure to define a recursive operator Ψ such that, for every f, if range$(f) = B$, then range$(\Psi(f)) = A$. On the other hand if $A \not\leq_e B$, then it is not difficult to construct a function g such that range$(g) = B$, but range$(\Psi_z(g)) \neq A$, all z, giving $\mathcal{E}_A \not\leq \mathcal{E}_B$. □

Remarkably, the degrees of solvability are definable in $\mathfrak{M}$, as is shown in the following theorem, which solves a problem raised in [10].

Theorem 2.3 **([2], [5])** *Let* $p(x) =^{df} (\exists y)[x < y \,\&\, (\forall z)[x < z \Rightarrow y \leq z]]$. *Then the degrees of solvability are exactly the degrees of difficulty satisfying* $p(x)$.

Proof. ($\Rightarrow$ ([5]):) Given a degree of solvability $\mathbf{S} = [\{f\}]$, let $\mathbf{S}' = [\{n * g : \Psi_n(g) = f \,\&\, g \not\leq_T f\}]$. Then $\mathbf{S} < \mathbf{S}'$ and $(\forall \mathbf{A})[\mathbf{S} < \mathbf{A} \Rightarrow \mathbf{S}' \leq \mathbf{A}]$.

($\Leftarrow$ ([2]):) Suppose that $\mathbf{A}$ is not a degree of solvability. If $\mathbf{A}$ is finite then clearly $\mathbf{A}$ does not satisfy $p(x)$, since $\mathbf{A}$ is easily seen to be meet–reducible. If $\mathbf{A}$ is not finite and $\mathcal{A} \in \mathbf{A}$, then for every mass problem $\mathcal{B}$ such that $\mathcal{B} \not\leq \mathcal{A}$, we construct a mass problem $\mathcal{C}$ such that $\mathcal{A} < \mathcal{C}$ and $\mathcal{B} \not\leq \mathcal{C}$. Here is a sketch of the construction.

(1) In order to get $\mathcal{A} \leq \mathcal{C}$, build $\mathcal{C}$ of the form $\mathcal{C} = \{x_n * f_n : n \in \omega\}$, where $f_n \in \mathcal{A}$; ensure also that $x_n \in \{0,1\}$ and $n \neq m \Rightarrow f_n \neq f_m$.

(2) Ensure that $\mathcal{C}$ satisfies the requirements:

- $P_e : \Psi_e(\mathcal{C}) \not\subseteq \mathcal{B}$;
- $R_e : \Psi_e(\mathcal{A}) \not\subseteq \mathcal{C}$.

Given any mass problem $\mathcal{X}$, let $\mathcal{X}^- = \{f^- : f \in \mathcal{X}\}$ (where, for every x, $f^-(x) = f(x+1)$). The mass problem $\mathcal{C}$ will be of the form $\mathcal{C} = \bigcup\{\mathcal{C}_n : n \in \omega\}$: let $\mathcal{C}_{-1} = \emptyset$.

Step $2e$) (Requirement P_e) Choose $x_{2e} \in \{0,1\}$ and $f_{2e} \in \mathcal{A}$ such that $f_{2e} \notin \mathcal{C}_{2e-1}^-$ and $\Psi_e(\mathcal{C}_{2e-1} \cup \{x_{2e} * f_{2e}\}) \not\subseteq \mathcal{B}$. Failure to find x_{2e} and f_{2e} would result in having $\Psi_e(\mathcal{D}) \subseteq \mathcal{B}$, where

$$\mathcal{D} = \mathcal{C}_{2e-1} \cup \{x * f : x \in \{0,1\} \,\&\, f \in \mathcal{A} - \mathcal{C}_{2e-1}^-\},$$

whence $\mathcal{B} \leq \mathcal{D}$. On the other hand we have also that $\mathcal{D} \leq \mathcal{A}$, via the recursive operator Ψ given by

$$\Psi(f) = \begin{cases} x_i * f_i, & \text{if } (\exists i)[\tilde{f}_i \subseteq f] \\ 0 * f, & \text{otherwise} \end{cases}$$

where $\{\tilde{f}_i : i \leq 2e-1\}$ are initial segments such that $S_{\tilde{f}_i} \cap \mathcal{C}_{2e-1}^- = \{f_i\}$ (use that the f_i's are distinct; here, given $\tilde{f}$, we let $S_{\tilde{f}} = \{f : \tilde{f} \subseteq f\}$). Finally, let $\mathcal{C}_{2e} = \mathcal{C}_{2e-1} \cup \{x_{2e} * f_{2e}\}$.

Step $2e+1$) (Requirement R_e) Notice that we can not have

$$(\forall f \in \mathcal{A})(\exists g \in \mathcal{C}_{2e-1}^{-})(\exists x \in \{0,1\})[\Psi_e(f) = x * g],$$

since otherwise we would obtain that $\mathcal{A} \equiv 0 * \mathcal{C}_{2e-1}^{-} \cup 1 * \mathcal{C}_{2e-1}^{-}$, which would imply that **A** is finite, contrary to the assumptions. Thus there exists a function $f \in \mathcal{A}$, such that $(\forall g \in \mathcal{C}_{2e-1}^{-})(\forall x \in \{0,1\})[\Psi_e(f) \neq x * g]$. Fix such an f: if $\Psi_e(f) = x * g$, for some $g \in \mathcal{A} - \mathcal{C}_{2e-1}^{-}$ and $x \in \{0,1\}$, then let $x_{2e+1} = 1 - x$ and $f_{2e+1} = g$; otherwise let e.g. $x_{2e+1} = 0$, and let f_{2e+1} be any function such that $f_{2e+1} \in \mathcal{A} - \mathcal{C}_{2e-1}^{-}$. Let $\mathcal{C}_{2e+1} = \mathcal{C}_{2e} \cup \{x_{2e+1} * f_{2e+1}\}$. □

Definition 2.4 For any degree of solvability **S**, let $\mathbf{S}' = \text{least } \{\mathbf{A} : \mathbf{S} < \mathbf{A}\}$.

Notice that $\mathbf{0}' = [\mathcal{O}]$, where $\mathcal{O} = \{f : f \text{ nonrecursive }\}$.

Problem 2.5 (**[10]**) *Are the degrees of enumerability definable? Or, at least, is the property of being a degree of enumerability a lattice–theoretic property?*

3 Lattice–theoretic properties

As remarked in [10] little is known about lattice theoretic properties of $\mathfrak{M}$. The following are very natural questions.

Problem 3.1 (**[10]**) *Is $\mathfrak{M}$ rigid?*

Problem 3.2 *Do the degrees of solvability constitute an automorphism basis for $\mathfrak{M}$?*

Problem 3.3 *Are the degrees of enumerability an automorphism basis for $\mathfrak{M}$?*

In [2], several lattice–theoretic properties are investigated.

Definition 3.4 Given mass problems $\mathcal{A}$ and $\mathcal{B}$ define $\mathcal{A} \leq_w \mathcal{B}$ if and only if $(\forall g \in \mathcal{B})(\exists f \in \mathcal{A})[f \leq_T g]$.

The preordering relation $\leq_w$ originates a partially ordered structure $\mathfrak{M}_w$, in the same way as $\leq$ originates $\mathfrak{M}$. The structure $\mathfrak{M}_w$ is in fact a complete distributive lattice, called the *Muchnik lattice*: see [7], [13]. For every mass problem $\mathcal{A}$, let $C(\mathcal{A}) = \{g : (\exists f \in \mathcal{A})[f \leq_T g]\}$. It is easy to see that $[\mathcal{A}]_w = [C(\mathcal{A})]_w$, (where $[\mathcal{B}]_w$ denotes the equivalence class of $\mathcal{B}$, for any mass problem $\mathcal{B}$). Also: $C(\mathcal{A}) \leq_w C(\mathcal{B}) \Leftrightarrow C(\mathcal{B}) \subseteq C(\mathcal{A})$.

Lemma 3.5 ([7],[13]) 1. *The mapping* $I : \mathfrak{M}_w \to \mathfrak{M}$ *defined by* $I([\mathcal{A}]_w) = [C(\mathcal{A})]$ *is an embedding preserving* $0, 1, \vee$ *and lowest upper bounds of arbitrary families.*

2. *The mapping* $F : \mathfrak{M} \to \mathfrak{M}_w$ *defined by* $F([\mathcal{A}]) = [\mathcal{A}]_w$ *is an onto lattice–theoretic homomorphism.*

Proof. Straightforward. □

Let us say that a degree of difficulty is a *Muchnik degree of difficulty* if it is in the range of the embedding I of Lemma 3.5 1. Then

Theorem 3.6 ([2]) *The property of being a Muchnik degree of difficulty is lattice–theoretic.*

Proof. First notice that $\mathbf{A}$ is a Muchnik degree of difficulty if and only if it contains a mass problem $\mathcal{A}$ such that $C(\mathcal{A}) = \mathcal{A}$. It is now easy to see that $\mathbf{A}$ is the least degree of difficulty containing some mass problem $\mathcal{B}$, such that

$$(\forall\{f\})[\mathcal{B} \leq \{f\} \Rightarrow (\exists g \in \mathcal{A})[\{g\} \leq \{f\}]].$$

Finally use the fact that the degrees of solvability are mapped to degrees of solvability by any automorphism (see Theorem 2.3). □

In a similar way, we can show e.g. that the property of containing a mass problem of the form $\{f : f_0 \leq_T f\}$, for any function f_0, is lattice–theoretic, etc.

Problem 3.7 *Do the Muchnik degrees form an automorphism basis for* $\mathfrak{M}$*?*

4 The structure

4.1 Incomparability results

The Medvedev lattice is as big as it can be:

Theorem 4.1 ([8]) $\mathfrak{M}$ *has antichains of cardinality* $2^{2^{\aleph_0}}$.

Proof. Let $\mathcal{A} = \{f_i : i \in I\}$, with $|I| = 2^{\aleph_0}$, be such that $\{[f_i]_T : i \in I\}$ is an antichain of $\mathfrak{D}_T$. Given any $X \subseteq I$, let $\mathcal{A}_X = \{f_i : i \in X\}$. Then, for every $X, Y \subseteq I$ such that $X|Y$, we have $\mathcal{A}_X|\mathcal{A}_Y$. The result then follows from observing that there are subsets $J \subseteq P(I)$ such that $|J| = 2^{2^{\aleph_0}}$ and if $X, Y \in J$ and $X \neq Y$ then $X|Y$. □

There are however maximal antichains of two elements!

Example 4.2 For any function f, let $\mathcal{B}_f = \{g : g \not\leq_T f\}$, and let $\mathbf{B}_f = [\mathcal{B}_f]$. Then it is easy to see that $\{\mathbf{B}_f, [\{f\}]\}$ is a maximal antichain (in fact, $(\forall\mathbf{C})[\mathbf{B}_f \not\leq \mathbf{C} \Rightarrow \mathbf{C} \leq [\{f\}]]$. Notice also that $\mathbf{B}_f \wedge [\{f\}] = \max\{\mathbf{C} : \mathbf{C} < \mathbf{B}_f\}$.

Theorem 4.3 ([2]) *If* $\mathbf{A} \neq \mathbf{0}, \mathbf{0}', \mathbf{1}$*, then there is a countable* $\mathbf{B}$ *such that* $\mathbf{B}$ *is incomparable with* $\mathbf{A}$.

Proof. Let $\mathcal{A} \in \mathbf{A}$, where $\mathbf{A} \neq \mathbf{0}, \mathbf{0}', \mathbf{1}$. Then, for every n, there is some nonrecursive function f_n such that $\Psi_n(f_n) \notin \mathcal{A}$ (otherwise $\Psi_n(\mathcal{O}) \subseteq \mathcal{A}$, hence $\mathbf{A} \leq \mathbf{0}'$.) Let $\mathcal{B} = \{f_n : n \in \omega\}$. Then $\mathcal{A} \not\leq \mathcal{B}$. If $\mathcal{B} \leq \mathcal{A}$, then take a function f which is Turing incomparable with all the members of $\mathcal{B}$ and $\{f\} \not\leq \mathcal{A}$ (such an f exists, since there can be at most countably many functions h such that $\{h\} \leq \mathcal{A}$, being $\mathcal{A} \neq \emptyset$). If $\mathcal{A} \not\leq \{f\}$, then $\mathcal{A}|\{f\}$. Otherwise, $\mathcal{B} \not\leq \mathcal{A}$, i.e. $\mathcal{A}|\mathcal{B}$. □

The following result establishes a sufficient condition for extending countable antichains of degrees of difficulty.

Theorem 4.4 ([2]) *Let* $\{\mathbf{A}_n : n \in \omega\}$ *be such that* $\mathbf{0}' < \mathbf{A}_n < \mathbf{1}$*, and, for all* n*, no nonzero finite degree of difficulty is below* $\mathbf{A}_n$*. Then there is a countable* $\mathbf{B}$ *such that* $\mathbf{B}|\mathbf{A}_n$*, for all* n.

Proof. Let $\mathcal{A}_n \in \mathbf{A}_n$. Construct a countable mass problem $\mathcal{B} = \{f_n : n \in \omega\}$, and a sequence of functions $\{g_n : n \in \omega\}$ such that $\mathcal{B} \cap \{g_n : n \in \omega\} = \emptyset$. At step n we define f_n, g_n. Suppose $n = \langle i, j\rangle$. Let $f_n \notin \{g_0, \dots, g_{n-1}\}$ be a nonrecursive function such that $\Psi_i(f_n) \notin \mathcal{A}_j$ (hence we satisfy the requirement $\Psi_i(\mathcal{B}) \not\subseteq \mathcal{A}_j$): such a function exists since otherwise we would have $\mathcal{A}_j \leq \mathcal{O}$). Finally define g_n to be a any function such that if $\Psi_i(f)$ total for all $f \in \mathcal{A}_j$, then $g_n \in \Psi_i(\mathcal{A}_j)$ and $g_n \notin \{f_0, \dots, f_n\}$: such a function exists, since otherwise $\{f_0, \dots, f_n\} \leq \mathcal{A}_j$, via Ψ_i. Since $g_n \notin \mathcal{B}$, this guarantees that $\Psi_i(\mathcal{A}_j) \not\subseteq \mathcal{B}$. □

We can characterize the countable lattices which can be embedded in $\mathfrak{M}$. Given a partial order $\mathfrak{P}$, let $1 \oplus \mathfrak{P} \oplus 1$ be the partial order obtained by adding an element at the bottom and an element at the top of $\mathfrak{P}$, respectively.

Theorem 4.5 ([13]) *A countable distributive lattice* $\mathfrak{L}$ *with* $0, 1$ *is embeddable in* $\mathfrak{M}$ *if and only if* 0 *is meet-irreducible and* 1 *is join-irreducible.*

Proof. Clearly, if $\mathfrak{L}$ is embeddable in $\mathfrak{M}$ then its least element is meet–irreducible, and the greatest element is join–irreducible, since this holds of $\mathfrak{M}$ as well. To show sufficiency, let $\mathfrak{B} \subseteq \mathfrak{P}(\omega)$ be an atomless countable Boolean algebra, where $\mathfrak{P}(\omega)$ is the Boolean algebra of the subsets of ω. It is not difficult to see that we can find a family $\{g, g_n : n \in \omega\}$ of nonrecursive functions and a recursive operator Ψ such that, for all m, n,

1. $g(0) = 0$, $g_n(0) = n + 1$;
2. $m \neq n \Rightarrow g_m|_T g_n \;\&\; g_m \vee g_n \equiv_T g$;
3. $m \neq n \Rightarrow \Psi(g_m \vee g_n) = g$.

For every subset $X \subseteq \omega$ let

$$\mathcal{A}_X = \begin{cases} \{g_x : x \in X\}, & \text{if } X \neq \emptyset \\ \{g\}, & \text{otherwise} \end{cases}$$

Then $Y \subseteq X \Leftrightarrow \mathcal{A}_X \leq \mathcal{A}_Y$: it is now easy to see that the mapping $X \mapsto [\mathcal{A}_X]$ yields a lattice–theoretic embedding of the dual $\breve{\mathfrak{B}}$ of $\mathfrak{B}$ into $\mathfrak{M}$ mapping 0 into $[\{g_n : n \in \omega\}]$, and 1 into $[\{g\}]$. Then claim then follows from the well known fact that for every distributive lattice $\mathfrak{L}$, we have that $1 \oplus \mathfrak{L} \oplus 1$ is embeddable in $1 \oplus \breve{\mathfrak{B}} \oplus 1$. □

Remark 4.6 Dyment ([2]) gives several topological interpretations and refinements of some of the results reviewed in this section. For instance, it is shown that one can define a topology on the collections of mass problems, such that, for any nonsolvable $\mathcal{A}$, the class $\{\mathcal{B} : \mathcal{B} \leq \mathcal{A}\}$ is of first category; it is interesting to note that one can also define a topology with respect to which, if $\mathcal{A}$ is such that, for no dense $\mathcal{B}$, is $\mathcal{A} \leq \mathcal{B}$, then $\{\mathcal{B} : \mathcal{A} \leq \mathcal{B}\}$ is of first category.

4.2 Empty intervals

A fairly straightforward refinement of the proof of Theorem 2.3 leads to the following useful characterization of empty intervals of $\mathfrak{M}$.

Theorem 4.7 ([2]) *Given* $\mathbf{A}, \mathbf{B}$, *with* $\mathbf{A} < \mathbf{B}$, *we have that* $(\mathbf{A}, \mathbf{B}) = \emptyset$ *if and only if* $(\exists$ *degree of solvability* $\mathbf{S})[\mathbf{A} = \mathbf{B} \wedge \mathbf{S}\ \&\ \mathbf{B} \not\leq \mathbf{S}\ \&\ \mathbf{B} \leq \mathbf{S}']$.

Proof. Let $\mathbf{A} < \mathbf{B}$, and let $\mathcal{A} \in \mathbf{A}$, $\mathcal{B} \in \mathbf{B}$.

($\Rightarrow$:) Suppose $(\mathbf{A}, \mathbf{B}) = \emptyset$. We observe that if $\mathcal{A}$ is finite, then there exists $f \in \mathcal{A}$ such that $\mathcal{B} \not\leq \{f\}$; let $\mathcal{E} \in \mathbf{S}'$, where $\mathbf{S} = [\{f\}]$. Then $\mathcal{A} \leq \mathcal{B} \wedge \{f\} < \mathcal{B}$, whence $\mathcal{A} \equiv \mathcal{B} \wedge \{f\} < \mathcal{B}$. On the other hand, $\mathcal{B} \wedge \mathcal{E} \not\leq \mathcal{A}$, otherwise $\mathcal{B} \wedge \mathcal{E} \leq \{f\}$, but then $\mathcal{B} \leq \{f\}$ or $\mathcal{E} \leq \{f\}$, contradiction. Then $\mathcal{A} < \mathcal{B} \wedge \mathcal{E} \equiv \mathcal{B}$, hence $\mathcal{B} \leq \mathcal{E}$.

If $\mathbf{A}$ contains no finite mass problem, and no degree of solvability $\mathbf{S}$ exists such that $\mathbf{A} = \mathbf{B} \wedge \mathbf{S}$, $\mathbf{B} \not\leq \mathbf{S}$, $\mathbf{B} \leq \mathbf{S}'$, then an easy modification of the proof of Theorem 2.3 enables us to construct a mass problem $\mathcal{C}'$ such that, letting $\mathcal{C} = \mathcal{B} \wedge \mathcal{C}'$, we have that $\mathcal{A} < \mathcal{C} < \mathcal{B}$.

($\Leftarrow$:) Let $\mathbf{A} = \mathbf{B} \wedge \mathbf{S}\ \&\ \mathbf{B} \not\leq \mathbf{S}\ \&\ \mathbf{B} \leq \mathbf{S}'$, where $\mathbf{S}$ is a degree of solvability; let $\mathcal{S} \in \mathbf{S}$, and $\mathcal{E} \in \mathbf{S}'$. Suppose that $\mathcal{A} < \mathcal{C} < \mathcal{B}$; then $\mathcal{B} \wedge \mathcal{S} \leq \mathcal{C}$, via, say, some recursive operator Ψ. Since $\Psi(\mathcal{C}) \subseteq 0 * \mathcal{B} \cup 1 * \mathcal{S}$, there are $\mathcal{C}_0, \mathcal{C}_1 \subseteq \mathcal{C}$ such that $\Psi(\mathcal{C}_0) \subseteq 0 * \mathcal{B}$ and $\Psi(\mathcal{C}_1) \subseteq 1 * \mathcal{S}$; moreover $\mathcal{C}_0 \wedge \mathcal{C}_1 \leq \mathcal{C}$. We can not have $\mathcal{C}_1 \leq \mathcal{S}$ since, otherwise, we would have $\mathcal{C} \leq \mathcal{B} \wedge \mathcal{S}$. Hence $\mathcal{S} < \mathcal{C}_1$. It follows that $\mathcal{E} \leq \mathcal{C}_1$, hence $\mathcal{B} \leq \mathcal{C}_0 \wedge \mathcal{C}_1 \leq \mathcal{C}$, contradiction. □

Example 4.8 If $\mathcal{B}$ is countable, then for any $\mathcal{A}$, with $\mathcal{A} < \mathcal{B}$, there is some mass problem $\mathcal{C}$ such that $\mathcal{A} < \mathcal{C} < \mathcal{B}$: indeed, for every f such that $\mathcal{B} \not\leq \{f\}$, one can find a function g such that $\{f\} < \{g\}$, but $\mathcal{B} \not\leq \{g\}$, hence $[\mathcal{B}] \not\leq [\{f\}]'$.

4.3 Bounds of countable families

The Medvedev lattice is not a complete lattice as follows from the results of this subsection. Given a lattice $\mathfrak{L}$, we say that a set $X \subseteq \mathfrak{L}$ is *strongly* $\wedge-$*incomplete* if $(\forall Y \subseteq X)[Y \text{ finite } \Rightarrow (\exists x \in X)[\bigwedge Y \not\leq x]]$. Dually one defines the notion of a *strongly* $\vee-$*incomplete* set $X \subseteq \mathfrak{L}$.

Theorem 4.9 ([3]) *No countable strongly* $\wedge-$*incomplete collection of degrees of difficulty has greatest lower bound.*

Proof. Let $\{\mathbf{A}_n : n \in \omega\}$ be a strongly $\wedge-$incomplete countable collection of degrees of difficulty (for instance let $\mathbf{A}_n = [\{f_n\}]$, where the functions f_n's are pairwisely Turing incomparable). We claim that $\{\mathbf{A}_n : n \in \omega\}$ does not have greatest lower bound. Let $\mathcal{A}_n \in \mathbf{A}_n$, for every n, and let $\mathcal{B}$ be a mass problem such that $\mathcal{B} \leq \mathcal{A}_n$, each n. In order to show the claim, construct a mass problem $\mathcal{C}$ such that $\mathcal{C} \leq \mathcal{A}_n$, all n, and $\mathcal{C} \not\leq \mathcal{B}$. Construct $\mathcal{C}$ of the form $\mathcal{C} = \bigcup\{x_n * \mathcal{A}_n : n \in \omega\}$ (with $x_m \neq x_n$ if $m \neq n$), hence $\mathcal{C} \leq \mathcal{A}_n$, all n. We define by induction two sequences $\{x_n : n \in \omega\}$ and $\{k_n : n \in \omega\}$ of numbers. At step n, we define x_n so as to satisfy the requirement $\Psi_n(\mathcal{B}) \not\subseteq \mathcal{C}$. By assumptions, there exists $f \in \mathcal{B}$ such that $\Psi_n(f) \notin \bigcup\{x_i * \mathcal{A}_i : i \leq n\}$, since $[\bigcup\{x_i * \mathcal{A}_i : i \leq n\}] = \bigwedge_{i \leq n} \mathbf{A}_i$. Choose such a function f and let $k_n = \Psi_n(f)(0)$ if $\Psi_n(f)$ is total, $k_n = 0$ otherwise: finally choose $x_n > \max(\{x_i : i < n\} \cup \{k_i : i \leq n\}$. □

Definition 4.10 A mass problem $\mathcal{A}$ is *effectively discrete* if $(\forall f, g \in \mathcal{A})[f \neq g \Rightarrow f(0) \neq g(0)]$.

Theorem 4.11 ([3]) *No countable strongly* $\vee-$*incomplete collection of effectively discrete degrees of difficulty has lowest upper bound.*

Proof. Let $\{\mathbf{A}_n : n \in \omega\}$ be a countable strongly $\vee-$incomplete collection of degrees of difficulty and for each n let $\mathcal{A}_n \in \mathbf{A}_n$ be an effectively discrete mass problem and let $\mathcal{B}$ be such that, for all n, $\mathcal{A}_n \leq \mathcal{B}$. We want to construct a mass problem $\mathcal{C}$ such that, for all n, $\mathcal{A}_n \leq \mathcal{C}$, but $\mathcal{B} \not\leq \mathcal{C}$. In order to satisfy the requirement P_n: $\Psi_n(\mathcal{C}) \not\subseteq \mathcal{B}$, we define at step n a number k_n such that, if Ψ_{i_n} is a recursive operator such that $\Psi_{i_n}(\mathcal{B}) \subseteq \mathcal{A}_{k_n}$ then $\Psi_{i_n}\Psi_n(\mathcal{C}) \not\subseteq \mathcal{A}_{k_n}$. Given any initial segment $\tilde{h}$, and functions $f_0, \ldots, f_n$, let $\tilde{h}(f_0, \ldots, f_n) = \tilde{h} \cup \{(\langle x, i\rangle, f_i(x - x_i)) : x \geq x_i,\, i \leq n\}$, where for each $i \leq n$, $x_i = \text{least}\{x : \tilde{h}(\langle x, i\rangle) \text{not defined }\}$.

Step n) Define a number k_n, and an initial segment $\tilde{h}_n$ as follows (assume $\tilde{h}_{-1} = \emptyset$).

Let $k_n = \text{least}\{i : \mathcal{A}_i \not\leq \mathcal{A}_0 \vee \ldots \vee \mathcal{A}_{n-1}\}$ (let k_0 be the least number such that $\mathcal{A}_{k_0}$ is not solvable). Such a number exists since the family is strongly $\vee$−incomplete.

Case 1) $(\exists g_0 \in \mathcal{A}_0, \ldots, \exists g_{n-1} \in \mathcal{A}_{n-1})(\forall f)[f \supseteq \tilde{h}_{n-1}(g_0, \ldots, g_{n-1}) \Rightarrow \Psi_{i_n}\Psi_n(f)(0) \uparrow]$. Then in this case, let $\tilde{h}_n = \emptyset$: the requirement P_n is automatically satisfied, if we ensure that there is some $f \in \mathcal{C}$ such that $f \supseteq \tilde{h}_{n-1}(g_0, \ldots, g_{n-1})$.

Case 2) Otherwise. Then, for every $g_0 \in \mathcal{A}_0, \ldots, g_{n-1} \in \mathcal{A}_{n-1}$, there exists an initial segment $F(\tilde{h}_{n-1}, g_0, \ldots, g_{n-1})$ extending $\tilde{h}_{n-1}$ and compatible with $\tilde{h}_{n-1}(g_0, \ldots, g_{n-1})$ such that $\Psi_{i_n}\Psi_n(F(\tilde{h}_{n-1}, g_0, \ldots, g_{n-1}))(0) \downarrow$. Since $\mathcal{A}_{k_n} \not\leq \mathcal{A}_0 \vee \ldots \vee \mathcal{A}_{n-1}$, there exist functions $g_0 \in \mathcal{A}_0, \ldots, g_{n-1} \in \mathcal{A}_{n-1}$ such that $V \notin \mathcal{A}_{k_n}$, where $V = \bigcup\{\Psi_{i_n}\Psi_n(f) : f \supseteq \tilde{h}_{n-1}(g_0, \ldots, g_{n-1}) \cup F(\tilde{h}_{n-1}, g_0, \ldots, g_{n-1})\}$. Fix such functions $g_0, \ldots, g_{n-1}$. If $g \not\subseteq V$, for all $g \in \mathcal{A}_{k_n}$, then define $\tilde{h}_n = \tilde{h}_{n-1} \cup (F(\tilde{h}_{n-1}, g_0, \ldots, g_{n-1}) - \tilde{h}_{n-1}(g_0, \ldots, g_{n-1}))$. Otherwise, since $\mathcal{A}_{k_n}$ is effectively discrete and there exists exactly one pair $(0, x) \in V$, there exists also exactly one function $f_n \in \mathcal{A}_{k_n}$ such that $f_n \subseteq V$. Since $V \not\subseteq f_n$, let $\tilde{h}$ be an initial segment extending $F(\tilde{h}_{n-1}, g_0, \ldots, g_{n-1})$, compatible with $\tilde{h}_{n-1}(g_0, \ldots, g_{n-1})$, such that $\Psi_{i_n}\Psi_n(\tilde{h}) \not\subseteq f_n$. Finally let $\tilde{h}_n = \tilde{h}_{n-1} \cup (\tilde{h} - \tilde{h}_{n-1}(g_0, \ldots, g_{n-1}))$. It clearly follows that, for no function $f \supseteq \tilde{h}_n(g_0, \ldots, g_{n-1})$, can we have $\Psi_{i_n}\Psi_n(f) \in \mathcal{A}_{k_n}$.

Let now $\mathcal{C}_n = \{g : (\exists f_0 \in \mathcal{A}_0), \ldots, (\exists f_n \in \mathcal{A}_n)[g \supseteq \tilde{h}_n(f_0, \ldots, f_n)]\}$, and, finally, let $\mathcal{C} = \bigcap\{\mathcal{C}_n : n \in \omega\}$. It immediately follows from the construction that $\Psi_{i_n}\Psi_n(\mathcal{C}) \not\subseteq \mathcal{A}_n$. Also, one immediately sees that, for each n, there exists a recursive operator Ψ, such that, if $g \supseteq \tilde{h}_n(f_0, \ldots, f_n)$, then $\Psi(g) = f_n$: thus $\mathcal{A}_n \leq \mathcal{C}_n$, hence $\mathcal{A}_n \leq \mathcal{C}$, since $\mathcal{C}_n \subseteq \mathcal{C}$. □

Since the degrees of solvability are effectively discrete degrees, one obtains as a corollary of the previous theorem Spector's result on the nonexistence of lowest upper bounds of ascending sequences of Turing degrees. In a similar way it is possible to show

Corollary 4.12 *No countable strongly $\vee$−incomplete family of degrees of enumerability has lowest upper bound.*

Proof. Similar to the proof of Theorem 4.11. □

Remark 4.13 There exist however infinite families with nontrivial lowest upper bound, as follows from Lemma 3.5: any collection of Muchnik degrees of difficulty has lowest upper bound.

5 Irreducible elements

It is not difficult to characterize the meet-reducible elements.

Theorem 5.1 ([2]) *A degree of difficulty* **A** *is meet-reducible if and only if* **A** *contains a mass problem* $\mathcal{A}$ *such that there exists r.e. sets* V_0, V_1 *of initial segments such that*

- $(\forall f \in \mathcal{A})(\exists i \in \{0,1\})(\exists \tilde{f})[\tilde{f} \in V_i \ \&\ \tilde{f} \subseteq f]$;
- $\{f : (\exists \tilde{f})[\tilde{f} \in V_0 \ \&\ \tilde{f} \subseteq f]\} | \{f : (\exists \tilde{f})[\tilde{f} \in V_1 \ \&\ \tilde{f} \subseteq f]\}$.

Proof. ($\Leftarrow$:) Given V_0, V_1, let $\mathcal{A}_0 = \{f : (\exists \tilde{f})[\tilde{f} \in V_0 \ \&\ \tilde{f} \subseteq f]\}$, and $\mathcal{A}_1 = \{f : (\exists \tilde{f})[\tilde{f} \in V_1 \ \&\ \tilde{f} \subseteq f]\}$. Then it is easy to see that $\mathcal{A}_0 | \mathcal{A}_1$, and $\mathcal{A} \equiv \mathcal{A}_0 \wedge \mathcal{A}_1$.

($\Rightarrow$:) If $\mathbf{A} = \mathbf{A}_0 \wedge \mathbf{A}_1$, $\mathcal{A}_0 \in \mathbf{A}_0$, $\mathcal{A}_1 \in \mathbf{A}_1$, and Ψ is a recursive operator such that $\Psi(\mathcal{A}) \subseteq 0 * \mathcal{A}_0 \cup 1 * \mathcal{A}_1$, then the problem $0 * \mathcal{A}_0 \cup 1 * \mathcal{A}_1$ and the r.e. sets $V_0 = \{\tilde{f} : \Psi(\tilde{f})(0) = 0\}$, $V_1 = \{\tilde{f} : \Psi(\tilde{f})(0) = 1\}$ satisfy the claim. □

The previous theorem is a useful tool for testing if a given element is meet-irreducible. Let us say ([7]) that a mass problem $\mathcal{A}$ is *uniform* if

$$(\forall \tilde{f})[S_{\tilde{f}} \cap \mathcal{A} \neq \emptyset \Rightarrow \tilde{f} * \mathcal{A} \subseteq \mathcal{A}].$$

As an application of the previous theorem we have for instance (see [2])

Corollary 5.2 *Every uniform degree of difficulty is meet-irreducible. Hence every Muchnik degree of difficulty is meet-irreducible.*

Proof. If $\mathcal{A}$ is uniform and $\mathcal{B} \wedge \mathcal{C} \leq \mathcal{A}$ via some recursive operator Ψ then suppose that there exists some function $f \in \mathcal{A}$ such that $\Psi(f) \in 0 * \mathcal{B}$: it follows that, for some initial segment $\tilde{f} \subseteq f$, $\Psi(\tilde{f})(0) = 0$. But $\tilde{f} * \mathcal{A} \subseteq \mathcal{A}$, hence $\Psi(\tilde{f} * \mathcal{A}) \subseteq 0 * \mathcal{B}$. It easily follows that $\mathcal{B} \leq \mathcal{A}$. Thus either $\mathcal{B} \leq \mathcal{A}$ or $\mathcal{C} \leq \mathcal{A}$. □

We do not have characterizations of the join-irreducible elements. Examples of join-irreducible elements are provided by Example 4.2: the element $\mathbf{B}_f$ is join-irreducible for every function f. We also notice the following useful lemma.

Lemma 5.3 *If* $\mathcal{A}$ *is a mass problem such that there exist functions* $f \in \mathcal{A}, g_1, g_2 \notin C(\mathcal{A})$ *and* $g_1 |_T g_2$ *and* $f \leq_T g_1 \vee g_2$, *then the degree of difficulty* $[\mathcal{A}]$ *is join-reducible.*

Proof. Under the assumptions, it is easy to see that

$$[\mathcal{A}] = [\mathcal{A} \wedge \{g_1\}] \vee [\mathcal{A} \wedge \{g_1\}].$$

On the other hand, it is clear that $[\mathcal{A} \wedge \{g_1\} | [\mathcal{A} \wedge \{g_1\}]$. □

Problem 5.4 *Characterize the join-irreducible elements.*

6 More on the degrees of enumerability. The Dyment lattice

In this section we collect some observations on the degrees of enumerability. We then define the Dyment lattice which extends the Medvedev lattice in much the same way as the enumeration degrees extend the Turing degrees. We show that there is an adjunction between the two lattices.

6.1 Some properties of the degrees of enumerability

The following theorem characterizes the degrees of enumerability corresponding to quasi–minimal e–degrees.

Theorem 6.1 ([2]) *Let $\mathcal{A}$ be a nonsolvable mass problem, and let A be a set of quasi–minimal $e-$degree. If $\mathcal{A} \leq \mathcal{E}_A$ then $\mathcal{A}$ is not countable.*

Proof. Suppose that $[A]_e$ is quasi–minimal, and let $\mathcal{A} = \{f_n : n \in \omega\}$ be a countable and nonsolvable mass problem. It is not difficult, given any n, to find, by finite extensions, a function f such that range$(f) = A$ and $\Psi_n(f) \neq f_n$. Failure to find f for some n would result in giving $f_n \leq_e A$, a contradiction. □

The remaining results of this subsection are contained in [13].

Theorem 6.2 *Every degree of enumerability is meet–irreducible. Every nonzero degree of enumerability is join–reducible.*

Proof. It is easy to see that, for every set A, the problem $\mathcal{E}_A$ is uniform, so it follows from Corollary 5.2 that $\mathbf{E}_A$ is meet–irreducible. That every nonzero degree of enumerability is join–reducible, follows from Lemma 5.3. □

Theorem 6.3 *Let* $\mathbf{E}$ *be a degree of enumerability. Then* $(\forall \mathbf{B} > \mathbf{E})[(\mathbf{E}, \mathbf{B}) \neq \emptyset]$. *Moreover,* $(\forall \mathbf{B} < \mathbf{E})[(\mathbf{B}, \mathbf{E}) \neq \emptyset]$.

Proof. In order to show that $(\mathbf{E}, \mathbf{B}) \neq \emptyset$, for any $\mathbf{B} > \mathbf{E}$, use Theorem 4.7 and the fact that each degree of enumerability is meet–irreducible.

To show the other part, use again Theorem 4.7 and the following observation. Given any degree of enumerability $\mathbf{E}_A > \mathbf{0}$, we can show that for every degree of solvability $\mathbf{S}$, if $\mathbf{E}_A \not\leq \mathbf{S}$, then $\mathbf{E}_A \not\leq \mathbf{S}'$. To see this, it is enough to show that for every function f such that $A \not\leq_e f$ one can find a function g such that $f <_T g$ and $A \not\leq_e g$: failure to do this would result in $A \leq_e f$, a contradiction. □

6.2 The Dyment lattice

Let $\mathcal{P}$ denote the collection of all partial functions from ω to ω. A *mass problem of partial functions* is any subset $\mathcal{A} \subseteq \mathcal{P}$. Given mass problems of partial functions $\mathcal{A}, \mathcal{B}$, we say that $\mathcal{A}$ is *e–reducible to* $\mathcal{B}$ (notation: $\mathcal{A} \leq_e \mathcal{B}$: the context will always make clear whether the symbol $\leq_e$ is used to denote enumeration reducibility between sets of numbers, or the above given reducibility between mass problems of partial functions) if there exists a partial recursive operator Ω such that $(\forall \phi \in \mathcal{B})[\phi \in \text{domain}(\Omega)\ \&\ \Omega(\phi) \in \mathcal{A}]$.

This preordering relation originates a degree structure which is a distributive lattice with 0, 1 called the *Dyment lattice* (see [2] and [13]) and denoted by $\mathfrak{M}_e$. The members of $\mathfrak{M}_e$ are called *partial degrees of difficulty* ; $[\mathcal{A}]_e$ denotes the *partial degree of difficulty of* $\mathcal{A}$. The operations of $\mathfrak{M}_e$ are: $[\mathcal{A}]_e \wedge [\mathcal{B}]_e = [\mathcal{A} \wedge \mathcal{B}]_e$; $[\mathcal{A}]_e \vee [\mathcal{B}]_e = [\mathcal{A} \vee \mathcal{B}]_e$ (where the operations $\wedge, \vee$ on mass problems of partial functions are defined in a similar way as for mass problems); moreover $\mathbf{0}_e = [\{\phi : \phi \text{ partial recursive}\}]_e$, and $\mathbf{1}_e = [\emptyset]_e$ are the least element and the greatest element, respectively.

Definition 6.4 1. A *total degree of difficulty* is a partial degree of difficulty containing a mass problem consisting of total functions;

2. a *partial degree of enumerability* is a partial degree of difficulty of the form $[\{\phi\}]_e$, for some partial function ϕ.

Clearly there is an embedding of $\mathfrak{D}_e$ onto the partial degrees of enumerability: just view $\mathfrak{D}_e$ as the upper semilattice of the partial degrees (i.e. equivalence classes of partial functions: see [10]): we have $\phi \leq_e \psi \Leftrightarrow \{\phi\} \leq_e \{\psi\}$.

Theorem 6.5 *The partial degrees of enumerability are definable in $\mathfrak{M}_e$ by the formula $p(x)$ of Theorem 2.3.*

Proof. The proof is similar to that of Theorem 2.3, but it has some original features since it is no longer true, in $\mathfrak{M}_e$, that the finite partial degrees of difficulty that are not partial degrees of enumerability are meet–reducible. For instance, it is easy to find examples of partial functions ϕ_1, ϕ_2 such that $\phi_1 \subset \phi_2$, but $\phi_1 |_e \phi_2$: thus $\{\phi_1, \phi_2\} <_e \{\phi_1\} \wedge \{\phi_2\}$. □

Let $\iota : \mathfrak{M} \longrightarrow \mathfrak{M}_e$ be the natural embedding, i.e. $\iota([\mathcal{A}]) = [\mathcal{A}]_e$. Notice that $\text{range}(\iota) = \{\mathbf{A}_e : \mathbf{A}_e \text{ total}\}$. Now, given a mass problem of partial functions $\mathcal{A}$, let

$$\mathcal{A}^* = \{f : (\exists \phi \in \mathcal{A})[\text{range}(f) = \text{graph}(\phi)]\}.$$

Let $\epsilon : \mathfrak{M}_e \longrightarrow \mathfrak{M}$ be defined by $\epsilon([\mathcal{A}]_e) = [\mathcal{A}^*]$: it is not difficult to see that ϵ is well defined and it is in fact ([13]) an onto lattice–theoretic homomorphism.

Theorem 6.6 ([2]) *For all $\mathbf{A}_e \in \mathfrak{M}_e$, and $\mathbf{B} \in \mathfrak{M}$ we have: $\mathbf{A}_e \leq_e \iota(\mathbf{B}) \Leftrightarrow \epsilon(\mathbf{A}_e) \leq \mathbf{B}$.*

Proof. It is not difficult to show that, for every $\mathcal{A}$,

$$[\mathcal{A}^*]_e = \text{least}\{\mathbf{B}_e : \mathbf{B}_e \text{ total } \& \ [\mathcal{A}]_e \leq_e \mathbf{B}_e\}. \square$$

For every partial function ϕ, let $\mathbf{E}_e^\phi = \iota(\epsilon([\{\phi\}]_e))$. $\mathbf{E}_e^\phi$ is called the *total degree of enumerability of* ϕ.

Corollary 6.7 ([2]) *The property of being a total degree of enumerability is invariant under all automorphisms F of $\mathfrak{M}_e$ such that $F(\iota(\mathfrak{M})) \subseteq \iota(\mathfrak{M})$.*

Proof. Immediate from the previous theorem and Theorem 6.5. $\square$

7 More on the Turing degrees

In this subsection we make some remarks on degrees of difficulty of the form $[C(\{g\})]$, for some function g. Very useful is the following lemma.

Lemma 7.1 ([2]) *If $\mathcal{A}$ is discrete and not solvable, then for every $\mathcal{B}$,*

$$\mathcal{A} \leq \mathcal{B} \Rightarrow \mathcal{B} \textit{ nowhere dense .}$$

Proof. Let $\mathcal{A}$, $\mathcal{B}$ be given, and let Ψ be such that $\Psi(\mathcal{B}) \subseteq \mathcal{A}$. In order to prove the claim, it is enough to show that $\mathcal{X} = \{f : \Psi(f) \in \mathcal{A}\}$ is nowhere dense. Otherwise there would be an initial segment $\tilde{f}_0$ such that $\mathcal{X}$ is dense in $S_{\tilde{f}_0}$ (we recall that the sets of the form $S_{\tilde{f}}$ are a basis for the Baire topology). Let $f_0 \supset \tilde{f}_0$ be such that $\Psi(f_0) \in \mathcal{A}$: since $\mathcal{A}$ is discrete, let $\tilde{g}_0$ be such that $S_{\tilde{g}_0} \cap \mathcal{A} = \{\Psi(f_0)\}$; thus, let $\tilde{f}_1 \supseteq \tilde{f}_0$ be such that $(\forall g \supset \tilde{f}_1)[\Psi(g) \in \mathcal{A} \Rightarrow \Psi(g) = \Psi(f_0)]$. Then $\Psi(f_0) = \bigcup\{\Psi(\tilde{g}) : \tilde{g} \supseteq \tilde{f}_1\}$, giving $\Psi(f_0)$ recursive, i.e. $\mathcal{A}$ solvable, a contradiction. $\square$

Theorem 7.2 *If $\mathcal{A}$ is not solvable and countable, then for every function g, $\mathcal{A} \not\leq C(\{g\})$.*

Proof. Let $\mathcal{A} = \{f_i : i \in \omega\}$ be nonsolvable, and let $\mathcal{B} = C(\{g\})$. One can show that for every n, if $\Psi_n(f)$ is total for every $f \in \mathcal{B}$, then there exists some function $f \in \mathcal{B}$ such that $(\forall i)[\Psi_n(f) \neq f_i]$: construct such an f of the form $f = h \vee g$, where $h = \bigcup\{\tilde{h}_i : i \in \omega\}$, and $\{\tilde{h}_i : i \in \omega\}$ is an increasing sequence of initial segments: failure at step i to find an initial segment $\tilde{h}_i$ such that $\Psi_n(\tilde{h}_i \vee g) \not\subseteq f_i$ would result in getting $\Psi_n(\{f \vee g : f \supset \tilde{h}_{i-1}\}) \subseteq \{f_i\}$, which, by the previous lemma, would imply $\{f \vee g : f \supset \tilde{h}_{i-1}\}$ nowhere dense, a contradiction (as usual, assume that $\tilde{h}_{-1} = \emptyset$). $\square$

Refinements of the previous theorem give:

Theorem 7.3 *For every function g, let g' be a function belonging to the Turing jump of $[g]_T$. Then*

1. *if $\mathcal{A}$ is not solvable and countable then $\mathcal{A} \not\leq \{f : g <_T f\}$.*

2. *$\{f : f$ nonrecursive $\&$ $f \leq_T g\} \not\leq \{f : g \leq_T f \leq_T g'\}$; $\{f : f$ nonrecursive $\&$ $f \leq_T g\} \not\leq \{f : f \equiv_T g'\}$; if g is not recursive then $\{f : f \equiv_T g\} | \{f : f \equiv_T g'\}$;*

3. *if $\mathcal{A}$ is not solvable and countable and $\mathcal{A} \not\leq \{h\}$, then $\mathcal{A} \not\leq C(\{g\}) \vee \{h\}$; hence if g is not recursive and does not have minimal Turing degree, then $\{f : f \equiv_T g\} \not\leq C(\{g\}) \vee \{f : f$ nonrecursive $\&$ $f \leq_T g\}$.*

Proof. For (1) modify step n in the proof of the previous theorem so as to construct $f = h \vee g'$. Inspection of the proof of the previous theorem shows that we can construct f such that $f \leq_T g'$, so (2) easily follows. For (3) relativize the proof of the previous theorem to h. □

8 Filters and ideals

Very little is known about nonprincipal filters and ideals of the Medvedev lattice (see [10]). We review some filters and ideals introduced in [2] and [12]. The results of this section, unless otherwise specified, can be found in [12].

Given any collection X of degrees of difficulty, let F_X and I_X denote the filter and the ideal, respectively, generated by X.

Given a mass problem $\mathcal{A}$, let $\Gamma_{\mathcal{A}} = \{[f]_T : f \in \mathcal{A} \ \& \ (\forall g \in \mathcal{A})[g \not<_T f]\}$. We say that a mass problem $\mathcal{A}$ has *countable basis* if $\Gamma_{\mathcal{A}}$ is countable and $(\forall g \in \mathcal{A})(\exists f)[f \leq_T g \ \& \ [f]_T \in \Gamma_{\mathcal{A}}]$. We say that $\mathcal{A}$ has *generalized countable basis* if $\Gamma_{\mathcal{A}}$ is countable.

We now define several classes X of degrees of difficulty.

Definition 8.1 Let $Solv = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ degree of solvability $\}$; $En = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ degree of enumerability $\}$; $Dis = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ discrete$\}$; $Edis = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ effectively discrete $\}$; $Count = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ countable $\}$; $Cl = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ closed $\}$; $CB = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ has countable basis$\}$; $GCB = \{\mathbf{A} : \mathbf{A} \neq \mathbf{0} \ \& \ \mathbf{A}$ has generalized countable basis $\}$; $D = \{\mathbf{A} : \mathbf{A}$ dense$\}$.

Since $\mathbf{0} \notin X$, all such X with $X \neq D$, we have that F_X is proper. Since $\mathbf{1} \notin D$, we have that I_D is proper.

We observe that if $X \in \{Dis, Edis, Count, CB, GCB, Cl\}$ then X is in fact a sublattice of $\mathfrak{M}$. Thus, in this case, $F_X = \{\mathbf{A} : (\exists \mathbf{B} \in X)[\mathbf{B} \leq \mathbf{A}]\}$.

Theorem 8.2 *The following hold: $F_{Solv} \subset F_{Edis} \subset F_{Dis} \subset F_{Count} \subset F_{GCB}$. Moreover $F_{Edis} \not\subseteq F_{CB}$; $F_{CB} \not\subseteq F_{Count}$; $F_{CB} \subset F_{GCB}$; $F_{Solv} \subset F_{CB}$; $F_{Edis} \subset F_{Cl}$; $F_{Dis} \not\subseteq F_{Cl}$.*

Proof. It is simple to show that the various inclusions hold. As to show that we have proper inclusions, or that some inclusions do not hold we can argue as follows. To show that $F_{Count} \not\subseteq F_{Dis}$, $F_{CB} \not\subseteq F_{Solv}$, use Lemma 7.1. To show that $F_{CB} \not\subseteq F_{Count}$ use Theorem 7.2. To show that $F_{Edis} \not\subseteq F_{Solv}$ consider as a counterexample the degree of difficulty of any effectively discrete mass problem consisting of two functions whose T–degrees constitute a minimal pair. To show that $F_{GCB} \not\subseteq F_{CB}$, consider the mass problem $\mathcal{A} = \{f : \neg(\exists g \leq_T f)[[g]_T \text{ minimal}]\}$. To show that $F_{Dis} \not\subseteq F_{Cl}$, construct (see [1]) a discrete and not solvable mass problem $\mathcal{A}$ such that, for every n, *either* there exists a function $f \in \mathcal{A}$ such that $\Psi_n(f)$ is not total, *or*, otherwise, there exists a recursive limit point g of $\mathcal{A}$, such that $\Psi_n(g)$ is total, so that, for every $\mathcal{B}$, if $\Psi_n(\mathcal{A}) \subseteq \mathcal{B}$ and $\mathcal{B}$ is closed, then $\mathcal{B}$ is solvable. To show that $F_{Edis} \not\subseteq F_{CB}$, consider the degree of difficulty of any nonsolvable and effectively discrete mass problem consisting of functions whose T–degrees, plus $\mathbf{0}_T$, constitute a densely and linearly ordered ideal of $\mathfrak{D}_T$. □

Theorem 8.3 *$F_{Solv} \subset F_{En}$; $F_{Edis} \not\subseteq F_{En}$; $F_{En} \not\subseteq F_{CB}$.*

Proof. $F_{En} \not\subseteq F_{Solv}$ follows from the existence of quasi–minimal e– degrees. To show $F_{Edis} \not\subseteq F_{En}$, take as a counterexample the degree of difficulty of any effectively discrete mass problem of functions whose e–degrees constitute pairs of minimal pairs in the e–degrees. Finally, $F_{En} \not\subseteq F_{CB}$ follows from the proof of Theorem 6.1. □

The above filters and ideals are all nonprincipal.

Theorem 8.4 *If $X \in \{Solv, En, Dis, Edis, Count, CB, GCB, Cl\}$ then F_X is nonprincipal. Moreover, the ideal I_D is nonprincipal.*

Proof. Let $X \in \{Dis, Edis, Count, CB, GCB\}$. To show that F_X is nonprincipal, it suffices to show that, for every $\mathbf{B} \in X$, there exists $\mathbf{A} \in F_X$ such that $\mathbf{B} \not\leq \mathbf{A}$. Given any $\mathbf{B} \in X$, find a degree of solvability $\mathbf{S}$ such that $\mathbf{B} \not\leq \mathbf{S}$: this follows directly by known incomparability results for the T–degrees. For $X = Cl$ see [1]. For $X = Solv$ and $X = En$, use also that the members of X are meet–irreducible. □

For most of the filters described above, the cardinality of the corresponding quotient lattice is determined by the following theorem.

Theorem 8.5 *The cardinalities of $\mathfrak{M}/F_{GCB}$ and $\mathfrak{M}/F_{En}$ are $2^{2^{\aleph_0}}$.*

Proof. Let $\{f_{(x,y)} : (x,y) \in I^2\}$ be a collection of functions whose T–degrees are minimal and constitute an antichain (here I denotes a set of cardinality $2^{\aleph_0}$). For every set $A \subseteq I$, let $\mathcal{A}_A = \{g : (\exists x \in A)(\exists y \in I)[f_{(x,y)} \leq_T g]\}$ (notice that the cardinality of $\Gamma_{\mathcal{A}_A}$ is $2^{\aleph_0}$). It is easy to see, using Corollary 5.2, that if $A \not\subseteq B$ then $[[\mathcal{A}_B]]_{GCB} \not\leq [[\mathcal{A}_A]]_{GCB}$ (where, given a degree of difficulty $\mathbf{A}$, the symbol $[\mathbf{A}]_{GCB}$ denotes its equivalence class in the quotient lattice $\mathfrak{M}/F_{GCB}$). This shows that $\mathfrak{M}/F_{GCB}$ has cardinality $2^{2^{\aleph_0}}$.

As to show that the cardinality of $\mathfrak{M}/F_{En}$ is $2^{2^{\aleph_0}}$, consider a function f and a collection $\{g_x : x \in I\}$ of functions whose T-degrees are an antichain, and $([f]_e, [g_x]_e)$ is a minimal pair of enumeration degrees, for all x. Then the collection $\{[[\mathcal{A}_A]]_{En} : A \subseteq I\}$, where $\mathcal{A}_A = \{g : f \leq_T g \vee (\exists i \in A)[g_i \leq_T g]\}$, has cardinality $2^{2^{\aleph_0}}$ in the quotient lattice. □

Problem 8.6 *Show that the cardinality of* $\mathfrak{M}/I_D$ *is* $2^{2^{\aleph_0}}$.

An important topic when we study a lattice is the investigation of its prime filters and prime ideals. A trivial example of a nonprincipal prime filter is $\mathfrak{M} - \{\mathbf{0}, \mathbf{0}'\}$. A trivial example of a nonprincipal prime ideal is $\mathfrak{M} - \{\mathbf{1}\}$. A more interesting example is given by:

Theorem 8.7 I_D *is prime.*

Proof. Suppose that $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}$ are mass problem such that $\mathcal{D}$ is dense and $\mathcal{D}_0 \wedge \mathcal{D}_1 \leq \mathcal{D}$, via, say the recursive operator Ψ. Then *either* $\Psi(\mathcal{D}) \subseteq 0 * \mathcal{D}_0$ *or* there exists an initial segment $\tilde{f}$ such that $\Psi(\tilde{f})(0) = 1$, and therefore $1 * \mathcal{D}_1 \leq \{f : \tilde{f} * f \in \mathcal{D}\}$, via the recursive operator $\lambda f.\Psi(\tilde{f} * f)$. Since $\{f : \tilde{f} * f \in \mathcal{D}\}$ is dense, the proof is complete. □

Let $F = \mathfrak{M} - I_D$. It follows from lattice theory that F is a prime filter. It is shown in [1] that F is nonprincipal and $|\mathfrak{M}/F| = 2^{2^{\aleph_0}}$, and $F_{Cl} \subsetneq F$. It follows also from next theorem that $F_{Dis} \subset F$ (hence $F_{Cl} \subset F$).

Theorem 8.8 *If* $X \in \{Solv,\ En,\ Edis,\ Dis,\ Count,\ CB,\ GCB\}$, *then* F_X *is not prime.*

Proof. Let f, B_0, B_1 be such that $f \leq_e B_0 \oplus B_1$, where B_0, B_1 belong to quasi-minimal e-degrees. Then $\mathbf{E}_{B_0}, \mathbf{E}_{B_1} \notin F_{CB}$ (this follows from Theorem 6.1), but $\mathbf{E}_{B_0} \vee \mathbf{E}_{B_1} \in F_{Solv}$. This shows that F_X is not prime if $X \in \{Solv,\ Edis,\ Dis,\ Count,\ CB\}$. As to show that F_{GCB} is not prime, consider a T-degree $\mathbf{a}$ and two families of minimal T-degrees R, S such that $|R| = |S| = 2^{\aleph_0}$ and $(\forall \mathbf{r} \in R)(\forall \mathbf{s} \in S)[\mathbf{a} \leq \mathbf{r} \vee \mathbf{s}]$ (in fact, for every $\mathbf{a}$, one can find such families). Then $\mathbf{A}_R (= [\{g : (\exists f)[[f]_T \in R \ \&\ f \leq_T g\}]) \notin F_{GCB}$, and $\mathbf{A}_S (= [\{g : (\exists f)[[f]_T \in S \ \&\ f \leq_T g\}]) \notin F_{GCB}$, but $\mathbf{A}_R \vee \mathbf{A}_S \in F_{GCB}$, in fact $\mathbf{A}_R \vee \mathbf{A}_S \in F_{CB}$.

Finally, to show that F_{En} is not prime, use the following fact about e-degrees (below, $A^{[i]}$ denotes the i^{th}- column of A): for every non r.e. set B, there exist a set A and a recursive function f such that $B \not\leq_e A^{[n]}$, all n, and $B \leq_e A^{[2m]} \oplus A^{[2n+1]}$ (all m, n) via the enumeration operator $\Phi_{f(m,n)}$, and $([A^{[2m]}]_e, [A^{[2n+1]}]_e)$ is a minimal pair in the e-degrees. Then it is easy to define two degrees of difficulty $\mathbf{A}, \mathbf{B}$, such that $\mathbf{A}, \mathbf{B} \notin F_{En}$, but $\mathbf{E}_B \leq \mathbf{A} \vee \mathbf{B}$. □

Problem 8.9 *Show that F_{Cl} is not prime.*

9 The Medvedev lattice as a Brouwer algebra

A distributive lattice $\mathfrak{L}$ with $0, 1$ is a *Brouwer algebra* if it can be equipped with a binary operation $\to$ such that, for all $a, b \in \mathfrak{L}$,

$$a \to b = \text{least}\{c \in \mathfrak{L} : b \leq a \vee c\}$$

(equivalently: $(\forall a, b, c)[b \leq a \vee c \Leftrightarrow a \to b \leq c]$). In the following, we will use the term *$B-$homomorphism* to denote any lattice-theoretic homomorphism which preserves $0, 1$ and $\to$.

We recall that a distributive lattice with $0, 1$ is a *Heyting algebra* if its dual is a Brouwer algebra.

Theorem 9.1 ([5]) *$\mathfrak{M}$ is a Brouwer algebra.*

Proof. Given mass problems $\mathcal{A}$, $\mathcal{B}$, define

$$\mathcal{A} \to \mathcal{B} = \{n * f : (\forall g \in \mathcal{A})[\Psi_n(g \vee f) \in \mathcal{B}]\}.$$

Then the following are easily seen:

- $\mathcal{B} \leq \mathcal{A} \vee (\mathcal{A} \to \mathcal{B})$, via Ψ, where $\Psi(g \vee (n * f)) = \Psi_n(g \vee f)$;
- $(\forall \mathcal{C})[\mathcal{B} \leq \mathcal{A} \vee \mathcal{C} \Rightarrow \mathcal{A} \to \mathcal{B} \leq \mathcal{C}]$: indeed, if $\Psi_n(\mathcal{A} \vee \mathcal{C}) \subseteq \mathcal{B}$ then let $\Psi'(f) = n * f$: clearly $\Psi'(\mathcal{C}) \subseteq \mathcal{A} \to \mathcal{B}$.

Thus, we have that $[\mathcal{A} \to \mathcal{B}] = \text{least}\{\mathbf{C} : [\mathcal{B}] \leq [\mathcal{A}] \vee \mathbf{C}\}$. □

Dyment (see [2]) defines a topology on the collection of mass problems, such that, if $\mathcal{B} \not\leq \mathcal{A}$, then $\{\mathcal{C} : \mathcal{A} \to \mathcal{B} \leq \mathcal{C}\}$ is of first category.

We notice however

Theorem 9.2 ([13]) *$\mathfrak{M}$ is not a Heyting algebra.*

Proof. It can be shown that, for every nonzero degree of solvability $\mathbf{S}$, there exists an effectively discrete degree $\mathbf{B}$ such that the set $\{\mathbf{C} : \mathbf{S} \wedge \mathbf{C} \leq \mathbf{B}\}$ does not have a greatest element: given a nonzero degree of solvability $\mathbf{S}$, let $\{g\} \in \mathbf{S}$. Construct a countable mass problem $\{f_n : n \in \omega\}$, such that, for every m, n with $m \neq n$,

- $g <_T f_n$ & $f_m(0) \neq f_n(0)$ & $f_m|_T f_n$;
- $\Psi_n(f_n) \neq g$.

Thus $\mathbf{B} = \{f_n : n \in \omega\}$ is the desired degree of difficulty. □

In the next theorem we characterize the finite Brouwer algebras that are B–embeddable in $\mathfrak{M}$. We will then discuss some of the consequences of this theorem. Let B' denote the class of Brouwer algebras in which 0 is meet–irreducible and 1 is join–irreducible.

Theorem 9.3 ([14]) *A finite Brouwer algebra $\mathfrak{L}$ is B–embeddable in $\mathfrak{M}$ if and only if $\mathfrak{L} \in B'$.*

Proof. Clearly, the condition $\mathfrak{L} \in B'$ is necessary for a finite Brouwer algebra to be B–embeddable in $\mathfrak{M}$: this follows from the fact that $\mathfrak{M} \in B'$.

We sketch the proof of the right–to–left implication. The proof is broken into several claims. Given any partial order $\mathfrak{P}$, let $F(\mathfrak{P})$ be the free distributive lattice with $0, 1$ generated by the partial order $\mathfrak{P}$ (hence $\mathfrak{P}$ embeds into $F(\mathfrak{P})$ as a partial order). It is not difficult to see:

Claim 1 *For any poset $\mathfrak{P}$, $F(\mathfrak{P})$ is a Brouwer algebra, in fact $F(\mathfrak{P}) \in B'$.*

We will now show that $F(\mathfrak{P})$ is B–embeddable in $\mathfrak{M}$, for a large class of posets $\mathfrak{P}$'s. In fact, $F(\mathfrak{D}_T)$ is B–embeddable in $\mathfrak{M}$.

Let now $\mathbf{2}$ be the two-element chain. Define B_J to be the smallest class of Brouwer algebras such that

1. $\mathbf{2} \in B_J$;
2. if $\mathfrak{L} \in B_J$ then $1 \oplus \mathfrak{L} \in B_J$;
3. B_J is closed under finite products.

Since a Brouwer algebra $\mathfrak{L}$ is subdirectly irreducible if and only if $\mathfrak{L} \simeq \mathbf{2}$ or $\mathfrak{L} \simeq 1 \oplus \mathfrak{L}'$, for some Brouwer algebra $\mathfrak{L}'$, it follows by the Birkhoff theorem on subdirectly irreducible algebras that, for every finite Brouwer algebra $\mathfrak{L}$, there exists $\mathfrak{L}' \in B_J$ such that $\mathfrak{L}$ is B–embeddable into $\mathfrak{L}'$. Thus, it is enough to show that for every $\mathfrak{L} \in B_J$, $1 \oplus \mathfrak{L} \oplus 1$ is B–embeddable in $\mathfrak{M}$.

We now notice

Claim 2 *For every $\mathfrak{L} \in B_J$, there exists a finite poset $\mathfrak{P}$ such that $1 \oplus \mathfrak{L} \oplus 1$ is B–embeddable in $F(\mathfrak{P})$.*

To see this, we argue by induction on the complexity of $\mathfrak{L}$ as a member of B_J, with respect to the three clauses of the definition of B_J. The most difficult part consists in showing that if $\{\mathfrak{L}_i : i \in I\}$ is a finite family for which we assume that, for all i, there exists a poset $\mathfrak{P}_i$ such that $1 \oplus \mathfrak{L}_i \oplus 1$ is B–embeddable in $F(\mathfrak{P}_i)$, then we can show that $1 \oplus (\prod \mathfrak{L}_i) \oplus 1$ is B–embeddable into $F(\coprod \mathfrak{P}_i)$, where $\coprod \mathfrak{P}_i$ denotes the coproduct of the $\mathfrak{P}_i$'s, in the category of partial orders.

Claim 3 *The Brouwer algebra $F(\mathfrak{D}_T)$ is $B-$embeddable into $\mathfrak{M}$.*

To prove the claim, let $\gamma : \mathcal{D}_T \longrightarrow \mathfrak{M}$ be defined by: $\gamma([f]_T) = \mathbf{B}_f$ (see Example 4.2). Let i be the lattice–theoretic homomorphism (preserving $0, 1$), $i : F(\mathfrak{D}_T) \longrightarrow \mathfrak{M}$, such that $i([f]_T) = \gamma([f]_T)$: such a homomorphism exists by the universal mapping property of $F(\mathfrak{D}_T)$ ($\mathcal{D}_T$ being a sub–poset of $F(\mathfrak{D}_T)$).

The proof of the claim is based on the fact (see Example 4.2 and Corollary 5.2) that each $\mathbf{B}_f$ is join–irreducible and meet–irreducible and that infima of finite collections of degrees of difficulty of the form $\mathbf{B}_f$ are dense and uniform, so that we can apply the following result, which, with routine calculations, implies that i preserves the Brouwer algebra operation $\rightarrow$.

Claim 4 *If $\mathbf{C}$ is dense and uniform then, for all $\mathbf{A}$, $\mathbf{B}$,*

$$\mathbf{C} \rightarrow \mathbf{A} \wedge \mathbf{B} = (\mathbf{C} \rightarrow \mathbf{A}) \wedge (\mathbf{B} \rightarrow \mathbf{C}).$$

In order to prove Claim 4, let $\mathcal{C}$ be a dense mass problem, and let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{X}$ be mass problems such that $\mathcal{A} \wedge \mathcal{B} \leq \mathcal{C} \vee \mathcal{X}$, via a recursive operator Ψ. For $i = 0, 1$, define

$$\mathcal{X}_i = \{f \in \mathcal{X} : (\exists \tilde{f})[\Psi(\tilde{f} \vee f)(0) \downarrow = i]\}.$$

Then $\mathcal{X} \equiv \mathcal{X}_0 \wedge \mathcal{X}_1$, and $\mathcal{A} \leq \mathcal{C} \vee \mathcal{X}_i$, for $i = 0, 1$. To show for instance that $\mathcal{A} \leq \mathcal{C} \vee \mathcal{X}_0$, one can use the recursive operator Ψ' defined informally as follows: in order to compute $\Psi'(f \vee g)$, given any enumeration of $f \vee g$, look for initial segments $\tilde{f}, \tilde{g}$ such that $\tilde{f} \subseteq f$ and $\tilde{g} \subseteq g$, and $\Psi(\tilde{f} \vee \tilde{g})(0) \downarrow = 0$; if no such initial segments are found then $\Psi'(f \vee g)$ is undefined, otherwise for the first such pair $\tilde{f}, \tilde{g}$ let $\Psi'(f \vee g) = \Psi((\tilde{f} * f) \vee g)$: density and uniformity of $\mathcal{C}$ ensure that $\tilde{f} * f \in \mathcal{C}$ if $f \in \mathcal{C}$, then $\Psi'(f \vee g) \in 0 * \mathcal{A}$ whenever $f \vee g \in \mathcal{C} \vee \mathcal{X}_0$, hence $\mathcal{A} \leq \mathcal{C} \vee \mathcal{X}_0$.

The following claim follows by easy calculations.

Claim 5 *If $\mathfrak{P}_1$, $\mathfrak{P}_2$ are posets and $\mathfrak{P}_1$ is order–theoretically embeddable into $\mathfrak{P}_2$, then $F(\mathfrak{P}_1)$ is $B-$embeddable into $F(\mathfrak{P}_1)$.*

We are now in a position to conclude the proof of the theorem. Since every finite partial order is embeddable into $\mathfrak{D}_T$, we have that for every Brouwer algebra $\mathfrak{L} \in B'$, there are (by Claim 2) a Brouwer algebra $\mathfrak{L}' \in B_J$, and a finite partial order $\mathfrak{P}$ such that $\mathfrak{L}$ is $B-$embeddable in $1 \oplus \mathfrak{L}' \oplus 1$, and $1 \oplus \mathfrak{L}' \oplus 1$ is $B-$embeddable in $F(\mathfrak{P})$. But, by Claim 5 $F(\mathfrak{P})$ is $B-$embeddable in $F(\mathfrak{D}_T)$. Finally the result follows from Claim 3. □

Interest in Brouwer (Heyting) algebras lies also in the fact that they are used for a semantics of certain intermediate logics. In the following we refer to a propositional language built up from an infinite countable set of propositional variables. Let *Form* denote the set of well-formed formulas. If $\mathfrak{L}$ is a Brouwer algebra, we say that a function $v : Form \longrightarrow \mathfrak{L}$ is an *$\mathfrak{L}$-valuation*, if for all $\alpha, \beta \in Form$, we have $v(\alpha \wedge \beta) = v(\alpha) \vee v(\beta)$, $v(\alpha \vee \beta) = v(\alpha) \wedge v(\beta)$, $v(\alpha \rightarrow \beta) = v(\alpha) \rightarrow v(\beta)$, $v(\neg\alpha) = v(\alpha) \rightarrow 1$ (where in the left hand side of these equations the symbols $\wedge, \vee, \rightarrow$ denote propositional connectives, and in the right hand side the same symbols denote operations of $\mathfrak{L}$).

Given any Brouwer algebra $\mathcal{L}$, and $\alpha \in Form$, let $\mathfrak{L} \models \alpha$ if $v(\alpha) = 0$, for all $\mathfrak{L}$-valuation v; finally let $Th(\mathfrak{L}) = \{\alpha \in Form : \mathfrak{L} \models \alpha\}$.

Let *Int*, *Class* denote the theorems of the intuitionistic propositional calculus and of the classical propositional calculus, respectively. Any deductively closed set $\Sigma \subseteq Form$ such that $Int \subseteq \Sigma \subseteq Class$ is called an *intermediate logic.* We are interested here in the intermediate logic *Jan* (after Jankov), i.e. the deductive closure of $Int \cup \{\neg\alpha \vee \neg\neg\alpha : \alpha \in Form\}$. We note the following result ([6]):

Corollary 9.4 $Th(\mathfrak{M}) = Jan$.

Proof. The result follows from Theorem 9.3 and the following observations: (1) if $\mathfrak{L}_1$ is B-embeddable in $\mathfrak{L}_2$, then $Th(\mathfrak{L}_2) \subseteq Th(\mathfrak{L}_1)$ (see e.g. [9]); (2) $Jan = \bigcap\{Th(\mathfrak{L}) : \mathfrak{L} \in B' \ \&\ \mathfrak{L} \text{ finite }\}$ (see [4]). On the other hand, simple calculations show that $Jan \subseteq Th(\mathfrak{M})$. □

Notice that, by Claim 3 and Claim 5 in the proof of Theorem 9.3, we can show that, if $\xi \leq 2^{\aleph_0}$ is a cardinal number, then the free distributive lattice with $0, 1$ on ξ generators is B-embeddable in $\mathfrak{M}$. More embeddings of Brouwer algebras will be pointed out in the proof of Theorem 9.10 below.

Problem 9.5 *Find examples of natural classes of infinite Brouwer algebras that are B-embeddable in $\mathfrak{M}$.*

Problem 9.6 *Show that for $\mathfrak{M}_e$ one can prove an embedding theorem similar to Theorem 9.3. What is $Th(\mathfrak{M}_e)$?*

It is well known that if $\mathfrak{L}$ is a Brouwer algebra and I is an ideal, then $\mathfrak{L}/I$ is still a Brouwer algebra. Clearly $Th(\mathfrak{L}) \subseteq Th(\mathfrak{L}/I)$, given the existence of the canonical onto homomorphism $\nu : \mathfrak{L} \longrightarrow \mathfrak{L}/I$. We recall

Theorem 9.7 ([15]) *If I is a proper principal ideal, then a finite Brouwer algebra $\mathfrak{L}$ is B-embeddable in $\mathfrak{M}/I$ if and only if $\mathfrak{L} \in B'$. It follows that $Th(\mathfrak{L}/I) = Jan$.*

Proof. It is easy to see that the proof of Theorem 9.3 relativizes to any proper principal ideal. □

Theorem 9.3 and the previous theorem make use of embeddings into $\mathfrak{M}$ whose ranges consist of dense degrees. It is therefore natural to ask the following question.

Problem 9.8 *Describe the finite Brouwer algebras that are $B-$embeddable in $\mathfrak{M}/I_D$. What is $Th(\mathfrak{M}/I_D)$?*

If G is a filter, then $\mathfrak{M}/G$ is not necessarily a Brouwer algebra. However, if G is a principal filter, then $\mathfrak{M}/G$ is a Brouwer algebra, as $\mathfrak{M}/G \simeq \{\mathbf{B} : \mathbf{B} \leq \mathbf{A}\}$, where $\mathbf{A}$ generates G.

Problem 9.9 *Study the set $\mathfrak{I} = \{Th(\mathfrak{M}/G) : G$ proper principal filter$\}$.*

We observe that $Class \in \mathfrak{I}$: just take the principal filter generated by $\mathbf{0}'$. Some remarks on this set can be found in [15].

We have the following remarkable result, due to Skvortsova.

Theorem 9.10 **([11])** *There exists a principal filter G such that $Th(\mathfrak{M}/G) = Int$.*

Proof. First show that if $\mathfrak{D}$ is any countable implicative uppersemilattice then $F(\mathfrak{D}) \oplus 1$ is $B-$embeddable in $\mathfrak{M}$, where $F(\mathfrak{D})$ is the free distributive lattice, generated by $\mathfrak{D}$ (i.e. $\mathfrak{D}$ embeds in $F(\mathfrak{D})$ via an embedding preserving $\vee, \rightarrow$ and 0). The embedding can be defined as follows. First, embed $F(\mathfrak{D})$ into $\mathfrak{D}_T$ as an initial segment (use for this the fact that $F(\mathfrak{D})$ is a countable distributive upper semilattice). For every $s \in F(\mathfrak{D})$, let f_s lie in the Turing degree corresponding to s under this embedding. Define

$$\mathcal{D}_s = \{f : f \text{ nonrecursive } \ \& \ (\forall s \in F(\mathfrak{D}))[f \not\equiv_T f_s]\} \cup \{f : f_s \leq_T f\}.$$

Then the embedding $\gamma : F(\mathfrak{D}) \oplus 1 \longrightarrow \mathfrak{M}$, given by $\gamma(s) = [\mathcal{D}_s]$ if $s \in F(\mathfrak{D})$, and $\gamma(1) = \mathbf{1}$, turns out to be a $B-$embedding (use the fact that every element in the range of i is the infimum of finitely many Muchnik degrees, hence Claim 4 of Theorem 9.3 applies).

Given any Brouwer algebra $\mathfrak{L}$ and $a, b \in \mathfrak{L}$, let $\mathfrak{L}_a = \{c \in \mathfrak{L} : c \leq a\}$, and $\mathfrak{L}_{a,b} = \{c \in \mathfrak{L} : a \leq c \leq b\}$. In both cases, we get a Brouwer algebra. If $c \in \mathfrak{L}$ is an element such that $c \vee a = b$ then there is an onto $B-$homomorphism $\nu : \mathfrak{L}_c \longrightarrow \mathfrak{L}_{a,b}$ (so $Th(\mathfrak{L}_c) \subseteq Th(\mathfrak{L}_{a,b})$). Now, let $\mathfrak{F}_\omega = \langle U, \vee, \rightarrow, 0\rangle$, with $U = \{X \subseteq \omega : X$ finite or cofinite$\}$, be the implicative upper semilattice where $X \vee Y = X \cap Y, X \rightarrow Y = X^c \cup Y$, and $0 = \omega$. Let $\delta : F(\mathfrak{F}_\omega) \oplus 1 \longrightarrow \mathfrak{M}$ be a $B-$embedding, constructed as above. It is possible ([11]) to find pairs $\{(a_i, b_i) : i \in \omega\}$ of elements of $F(\mathfrak{F}_\omega)$, such that $a_i \in \mathfrak{F}_\omega$, $a_i < b_i$, and $\bigcap Th(F((\mathfrak{F}_\omega)_{a_i,b_i}) : i \in \omega\} = Int$. For every i, let $b_i = \bigwedge\{b_i^j : j \leq n_i\}$, with $b_i^j \in \mathfrak{F}_\omega$, and, finally, let $\mathcal{E} = \{i * j * f : j \leq n_i, \ f \in \mathcal{D}_{b_i^j}\}$ (where $\mathcal{D}_{b_i^j} \in \delta(b_i^j)$). Let $\mathbf{E} = [\mathcal{E}]$, and, for all i, let $\mathbf{A}_i, \mathbf{B}_i$ be the images under δ of a_i, b_i, respectively. It is not difficult to see that for every i, $\mathbf{E} = \mathbf{A}_i \vee \mathbf{B}_i$. By the above remarks we have that $\bigcap\{Th(\mathfrak{M}_{\mathbf{A}_i,\mathbf{B}_i}) : i \in \omega\} = Int$, and thus $Th(\mathfrak{M}_\mathbf{E}) = Int$. □

It follows from a result in [11] that if $\mathfrak{H}$ is the collection of proper principal filters generated by elements that are infima of finitely many Muchnik degrees, then $Int = \bigcap\{(Th(\mathfrak{M}/G) : G \in \mathfrak{H}\}$.

Problem 9.11 ([11]) *Is it possible to find* $G \in \mathfrak{H}$ *such that* $Th(\mathfrak{M})/G = Int$*?*

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