Take-home final exam for Math 771: Set Theory; Spring 2016
Due: May 16 at noon

1) Show that ZF proves the following fact: if (M,R) is a model such that

$$(M,R) \models \mathrm{ZF} + \mathrm{V} = \mathrm{L},$$

and also such that R is well-founded on M, then there is an ordinal α such that (M,R) is isomorphic to $(L(\alpha),\in)$.

2) In this problem, we assume that Martin's Axiom is true, and that $2^{\aleph_0} \geq \aleph_{38}$. Let $\mathcal{A},\mathcal{B} \subseteq \mathcal{P}(\omega)$ both have size $\aleph_{37}$. Furthermore, let us assume that whenever $\mathcal{A}' \subseteq \mathcal{A}$ and $\mathcal{B}' \subseteq \mathcal{B}$ are such that $\mathcal{A}'$ and $\mathcal{B}'$ both have size $\aleph_1$, then there is a set $x \subseteq \omega$ satisfying that $\forall a \in \mathcal{A}'(a \subseteq^* x)$ and $\forall b \in \mathcal{B}'(b \perp x)$ (see Definition III.1.15). Show that there is an $x \subseteq \omega$ such that in fact $\forall a \in \mathcal{A}(a \subseteq^* x)$ and $\forall b \in \mathcal{B}(b \perp x)$.
Hint: consider

$$\mathbb{P} = \Big\{(X,Y) \mid X,Y \subseteq \omega, X \cap Y = \emptyset, \text{ and there exist finite subsets } U \subseteq \mathcal{A}, V \subseteq \mathcal{B} \text{ such that } X =^* \bigcup U \text{ and } Y =^* \bigcup V.\Big\}$$

3) Let M be a ctm for ZF. Let $N = (L)^M$. Let $\mathbb{P} = \mathrm{Fn}((\omega_2)^N \times \omega, \omega)$. Let G be $\mathbb{P}$-generic over N. We consider the model $N[G]$.

We will be showing, in steps, that $N[G]$ is a model of the statement that there is a sequence of ω_2 many functions $f_\gamma : \omega \to \omega$ such that for every $g \in \omega^\omega$, $\{\gamma \mid f_\gamma \leq^* g\}$ is countable.

In fact, we get these functions as follows: given $\gamma < (\omega^2)^N$, let $f_\gamma = \{(n,m) \mid \{((\gamma,n),m)\} \in G\}$.

a) Show that $N[G]$ satisfies the statement "every f_γ is a total function, and the set $\{f_\gamma \mid \gamma < (\omega_2)^N\}$ has size ω_2".

b) Let $f \in N[G]$ be such that $N[G] \models f : \omega \to \omega$. Show: there is set $\Gamma \subseteq (\omega_2)^N$ such that (Γ is countable)^N, and there is an $\mathring{f} \in N^{\mathbb{P}}$ satisfying $\mathring{f}_G = f$ such that $\mathring{f}$ is actually a $\mathrm{Fn}(\Gamma \times \omega, \omega)$-name.

c) Let $f \in N[G]$ be such that $N[G] \models f : \omega \to \omega$, and let Γ and $\mathring{f}$ be for f as in the previous subproblem. Show: if $\gamma \notin \Gamma$, then

$$N[G] \models f_\gamma \not\leq^* f.$$

Hint: let $(p_i)_{i<\omega} \in N$ be an enumeration of $\mathrm{Fn}(\Gamma \times \omega, \omega)$. Working in N, for every $i \in \omega$, if there is a $q_i \leq p_i$ with $q_i \in \mathrm{Fn}(\Gamma \times \omega, \omega)$ and an $n_i \in \omega$ such that

$$q_i \Vdash \mathring{f}(i) = \check{n}_i,$$

choose such a q_i and n_i. Now let D_γ be the set of those $r \in \mathbb{P}$ such that for some $i \in \omega$, both $r \leq q_i$ and $r \leq \{((\gamma,i),n_i+1)\}$. Argue that $G \cap D_\gamma \neq \emptyset$.

d) Show that $N[G]$ is indeed a model of the statement that there is a sequence of ω_2 many functions $f_\gamma : \omega \to \omega$ such that for every $g \in \omega^\omega$, $\{\gamma \mid f_\gamma \leq^* g\}$ is countable.