20 Shoenfield Absoluteness

For a tree $T \subseteq \bigcup_{n<\omega} \kappa^n \times \omega^n$ define

$$p[T] = \{y \in \omega^\omega : \exists x \in \kappa^\omega \;\; \forall n(x \upharpoonright n, y \upharpoonright n) \in T\}.$$

A set defined this way is called κ-Souslin. Thus $\mathbf{\Sigma}^1_1$ sets are precisely the ω-Souslin sets. Note that if $A \subseteq \omega^\omega \times \omega^\omega$ and $A = p[T]$ then the projection of A, $\{y : \exists x \in \omega^\omega \; (x,y) \in A\}$ is κ-Souslin. To see this let $<,>: \kappa \times \omega \to \kappa$ be a pairing function. For $s \in \kappa^n$ let $s_0 \in \kappa^n$ and $s_1 \in \omega^n$ be defined by $s(i) =< s_0(i), s_1(i) >$. Let T^* be the tree defined by

$$T^* = \bigcup_{n\in\omega} \{(s,t) \in \kappa^n \times \omega^n : (s_0, s_1, t) \in T\}.$$

Then $p[T^*] = \{y : \exists x \in \omega^\omega \; (x,y) \in A\}$.

Theorem 20.1 *(Shoenfield [96]) If A is a Σ^1_2 set, then A is ω_1-Souslin set coded in L, i.e. $A = p[T]$ where $T \in L$.*

proof:

From the construction of T^* it is clear that is enough to see this for A which is Π^1_1.

We know that a countable tree is well-founded iff there exists a rank function $r : T \to \omega_1$. Suppose

$$x \in A \text{ iff } \forall y \exists n \;\; (x \upharpoonright n, y \upharpoonright n) \notin T$$

where T is a recursive tree. So defining $T_x = \{t : (x \upharpoonright |t|, t) \in T\}$ we have that $x \in A$ iff T_x is well-founded (Theorem 17.4).

The ω_1 tree $\hat{T}$ is just the tree of partial rank functions. Let $\{s_n : n \in \omega\}$ be a recursive listing of $\omega^{<\omega}$ with $|s_n| \leq n$. Then for every $N < \omega$, and $(r,s) \in \omega_1^N \times \omega^N$ we have $(r,t) \in \hat{T}$ iff

$$\forall n,m < N \; [(t, s_n),(t, s_m) \in T \text{ and } s_n \subset s_m] \text{ implies } r(n) > r(m).$$

Then $A = p[\hat{T}]$. To see this, note that if $x \in A$, then T_x is well-founded and so it has a rank function and therefore there exists r with $(x,r) \in [\hat{T}]$ and so $x \in p[\hat{T}]$. On the other hand if $(x,r) \in [\hat{T}]$, then r determines a rank function on T_x and so T_x is well-founded and hence $x \in A$.

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Theorem 20.2 *(Shoenfield Absoluteness [96]) If $M \subseteq N$ are transitive models of ZFC* and $\omega_1{}^N \subseteq M$, then for any $\Sigma^1_2(x)$ sentence θ with parameter $x \in M$*

$$M \models \theta \textit{ iff } N \models \theta.$$

proof:

If $M \models \theta$, then $N \models \theta$, because $\underset{\sim}{\Pi}^1_1$ sentences are absolute. On the other hand suppose $N \models \theta$. Working in N using the proof of Theorem 20.1 we get a tree $T \subseteq \omega_1^{<\omega}$ with $T \in L[x]$ such that T is ill-founded, i.e., there exists $r \in [T]$. Note that r codes a witness to a $\Pi^1_1(x)$ predicate and a rank function showing the tree corresponding to this predicate is well-founded. Since for some $\alpha < \omega_1$, $r \in \alpha^\omega$ we see that

$$T_\alpha = T \cap \alpha^{<\omega}$$

is ill-founded. But $T_\alpha \in M$ (since by assumption $(\omega_1)^N \subseteq M$) and so by the absoluteness of well-founded trees, M thinks that T_α is ill-founded. But a branch thru $[T]$ gives a witness and a rank function showing that θ is true, and consequently, $M \models \theta$.

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